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THEORY OF RADIALLY STRATIFIED MEDIA

PART II. ASYMPTOTIC EXPANSION OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

By

John G. Fikioris

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Cruft Laboratory
Harvard University
Cambridge, Massachusetts

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PART II

ASYMPTOTIC EXPANSIONS OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

CHAPTER 1

SOLUTION OF THE RADIAL DIFFERENTIAL EQUATION FOR TM WAVES

CONVERGENT SERIES SOLUTIONS AROUND $x = 0$

We consider equation (1-50), PART I:

$$\frac{d^2R(x)}{dx^2} + \frac{c}{(x+a)(x+b)}\frac{dR(x)}{dx} + \left[1+\frac{c}{x+b} - \frac{v(v+1)}{x^2}\right] R(x) = 0 \quad .(\text{I } 1\text{-}50)$$

The origin $x=0$ is a regular singularity of this equation. In order to obtain convergent series solutions around this point, we follow the classical method of Frobenius (9 pp. 396-404). First, write the equation in the form:

$$[x^2+(a+b)x+ab]x^2 R''(x)+cx^2 R'(x)+[x^4+2ax^3+(a^2-v(v+1))x^2-v(v+1)\cdot$$

$$\cdot(a+b)x-v(v+1)ab] R(x) = 0 \quad .$$

Substituting the formal series $R(x) = \sum_{n=0}^{\infty} a_n x^{n+\sigma}$ the equation yields: $\sum_{n=0}^{\infty} a_n x^{n+\sigma} f(x,n+\sigma) = 0$, where:

$$f(x,n+\sigma)=(n+\sigma)(n+\sigma-1)[x^2+(a+b)x+ab]+(n+\sigma)cx+x^4+2ax^3+[a^2-v(v+1)]x^2-$$
$$-v(v+1)(a+b)x-v(v+1)ab = f_o(n+\sigma)+f_1(n+\sigma)x+f_2(n+\sigma)x^2+f_3(n+\sigma)x^3+$$
$$+f_4(n+\sigma)x^4.$$

The functions appearing in the last expression are defined as follows:

$$f_o(n+\sigma) = ab[(n+\sigma)(n+\sigma-1)-v(v+1)]=ab(n+\sigma+v)(n+\sigma-v-1) \qquad (1\text{-}1)$$

$$f_1(n+\sigma) = (a+b)(n+\sigma)(n+\sigma-1)+c(n+\sigma)-(a+b)v(v+1) \qquad (1\text{-}2)$$

$$f_2(n+\sigma) = (n+\sigma)(n+\sigma-1)+a^2-v(v+1) \qquad (1\text{-}3)$$

$$f_3(n+\sigma) = 2a \qquad (1\text{-}4)$$

$$f_4(n+\sigma) = 1 \quad . \qquad (1\text{-}5)$$

Indicial equation:

$$f_o(\sigma) = ab(\sigma+v)(\sigma-v-1) = 0 \qquad (1\text{-}6)$$

with roots:

$$\sigma_1= v+1 \quad , \quad \sigma_2= -v \quad . \qquad (1\text{-}7)$$

Reccurence formula for the coefficients:

$$a_nf_o(n+\sigma)+a_{n-1}f_1(n+\sigma-1)+a_{n-2}f_2(n+\sigma-2)+a_{n-3}f_3(n+\sigma-3)+$$
$$+a_{n-4}f_4(n+\sigma-4) = 0 \quad ; \quad a_{-m}= 0 \ , \ m = 1,2,3,.... \quad . \qquad (1\text{-}8)$$

v is a positive parameter, since negative values of v do not yield modes independent from the ones corresponding to positive values of v. We have:

$$\sigma_1- \sigma_2 = 2v+1 \quad . \qquad (1\text{-}9)$$

If $2v+1$ is not an integer, the method yields two independent solutions. In particular, we define as $R_1(x)$ and $R_2(x)$ the solutions corresponding to $\sigma_1 = v+1$ and $\sigma_2 = -v$, respectively, with $a_o = 1$, i.e.

$$R_1(x) = x^{v+1}[1+\sum_{n=1}^{\infty} a_n x^n] \quad , \quad |x| < \min(|a|,|b|) \qquad (1\text{-}10)$$

$$R_2(x) = x^{-v}[1+\sum_{n=1}^{\infty} b_n x^n] \quad , \quad |x| < \min(|a|,|b|) \quad , \quad 2v+1 \neq \text{integer} \qquad (1\text{-}11)$$

For the coefficients a_n we use (1-8) with $\sigma=\sigma_1=v+1$ and $a_o = 1$, for the b_n's, again (1-8) with $\sigma=\sigma_2=-v$ and $b_o = 1$. We can find that:

$$a_1 = -f_1(\sigma_1)/f_o(\sigma_1+1) = -\frac{c}{2ab} \, , \quad b_1 = -f_1(\sigma_2)/f_o(\sigma_2+1) = -\frac{c}{2ab} \, .$$

Also:

$$\min(|a|,|b|) = \begin{cases} |a| & \text{if } |a|<|b| \\ |b| & \text{if } |b|<|a| \end{cases} \, . \qquad (1\text{-}12)$$

We observe that $R_1(0) = 0$ and $R_2(0) = \infty$. $x=0$ is a branch point for both solutions.

If $2v+1$ is equal to a positive integer, (1-10), together with (1-8), continues to define $R_1(x)$. The second independent solution $R_2(x)$ in this case, has a logarithmic singularity at $x=0$ and is going to be found later.

<u>Analytic Continuation of $R_1(x)$, $R_2(x)$ in the Right-Half x-Plane:</u> We obtained two series representations for the solutions $R_1(x)$ and $R_2(x)$ of equation (I 1-50). They are valid only within the circle $|x|<\min(|a|,|b|)$, where they both converge uniformly and absolutely. We proceed to obtain representations for these particular functions valid in the whole right-half plane, where x varies from 0 to ∞.

We use a bilinear transformation of the independent variable in the form:

$$t = \frac{x}{x+p} \quad , \qquad x = \frac{pt}{1-t} \quad . \tag{1-13}$$

The constant p is conveniently chosen in each case so as to optimize the rate of convergence of the resulting series in t. The conformal mapping of x-plane on to t-plane, in accordance with (1-13), is shown in figure (1-1). The half-plane to the right of the perpendicular bisecting the line between 0 and -p (shaded in figure (1-1)), maps within the unit circle $|t| = 1$ of the t-plane. In this plane x varies from 0 to ∞ along a straight line in the fourth quadrant, or along the real axis. Correspondingly, t varies from 0 to 1 within the unit circle. The singular points of equation (I 1-50) map as follows:

$x = 0 \qquad t = 0$
$x = -a \qquad t = a/(a-p)$
$x = -b \qquad t = b/(b-p)$
$x = \infty \qquad t = 1 \quad .$

The equation becomes:

$$\frac{d^2R}{dt^2} + \left[\frac{pc}{[(p-a)t+a][(p-b)t+b]} + \frac{2}{t-1}\right]\frac{dR}{dt} +$$

$$+ \left[\frac{(p-a)t+a}{(p-b)t+b}\,\frac{p^2}{(t-1)^4} - \frac{v(v+1)}{t^2(t-1)^2}\right] R = 0 \quad . \tag{1-14}$$

It now has three regular singularities at $t=0$, $t=b/(b-p)$, $t=a/(a-p)$ and an irregular singularity at $t=1$. Thus, the transformation has preserved the nature and rank of the singularities, mapping them at their image points in the t-plane. This is a well-known property of bilinear transformations (9 p. 437). If p is chosen as shown in figure (1-1), the singularities $x=-a$ and $x=-b$ map outside the unit circle in the t-plane. A power series solution of equation (1-14) around $t=0$ will converge uniformly and absolutely for $|t|<1$ and would provide a representation valid

over the whole interval of interest $0 \leqslant \mathrm{Re}\,x \leqslant \infty$. The numerical computations of Chapter 3, PART I, showed that such series can be used for an $|x|_{max}$ 3 or 4 times larger than the $|x|_{max}$ of (1-10) and (1-11). It is not necessary to map $x=-a$ and $x=-b$ outside the circle $|t| = 1$. Depending on the case under consideration, a value of p can be chosen, that maps either or both of these points inside $|t| = 1$, but which yields better series in t. The radius of convergence is now: $|t| = \min\left(\left|\frac{a}{a-p}\right|, \left|\frac{b}{b-p}\right|\right) < 1$; nevertheless, larger values of x can be used with such series, that may carry one farther. In this connection, notice that x varies only along a straight line in the fourth quadrant, or along the real axis, from 0 to ∞ and that one is only interested in reaching up to such values of x after which the asymptotic expansions of $R_1(x)$ and $R_2(x)$ can be used. The fact that a series expansion can be found, which is valid for all x from 0 to ∞, has no practical significance from the computational point of view. A series valid within a smaller, finite region may be better if it can be used for larger values of x. These facts were verified numerically. $|p|$ should be chosen as large as possible, but not as large as to result in a very small convergence circle in the t-plane. In each particular case an optimum value of p exists.

For all Cases I to VI, Chapter 3, PART I, p was given the following value:

$$p = 2a \quad , \quad t = \frac{x}{x+2a} \quad , \quad x = \frac{2at}{1-t} \quad . \tag{1-15}$$

We will develop the series expansions in t for $R_1(x)$ and $R_2(x)$ for this particular choice. For a different p the analysis follows identical lines. Now, $x=-a$ maps on $t=-1$ and (1-14) becomes:

$$(t-1)^4(t+1)\left(t+\frac{b}{2a-b}\right)t^2R''(t)+2\left[\frac{c}{2a-b}(t-1)^4+(t-1)^3(t+1)\left(t+\frac{b}{2a-b}\right)\right]\cdot$$

$$\cdot t^2 R'(t)+\left[\frac{4a^3}{2a-b}(t+1)^2t^2-v(v+1)(t+1)\left(t+\frac{b}{2a-b}\right)\right]R(t) = 0 \ . \qquad (1\text{-}16)$$

Inserting: $R(t) = \sum_{n=o}^{\infty} e_n t^{n+\sigma}$ the equation becomes:

$\sum_{n=o}^{\infty} e_n t^{n+\sigma} f(t,n+\sigma) = 0$, where:

$$f(t,n+\sigma)=(n+\sigma)(n+\sigma-1)[t^6+(h-3)t^5+(2-3h)t^4+2(h+1)t^3+(2h-3)t^2+$$

$$+(1-3h)t+h]+(n+\sigma)[2t^6+(h-3)t^5-4t^4+(10-6h)t^3+(8h-6)t^2-(3h-1)t]+$$

$$+[2a^2(h+1)-v(v+1)]t^4+[4a^2(h+1)+(1-h)v(v+1)]t^3+(1+h)[2a^2+v(v+1)]t^2+$$

$$+(h-1)v(v+1)t-hv(v+1) = \sum_{m=o}^{6} f_m(n+\sigma)t^m \ .$$

The following definitions were used:

$$h = b/(2a-b) \qquad (1\text{-}17)$$

$$f_o(x) = h[x(x-1)-v(v+1)] = h(x+v)(x-v-1) \qquad (1\text{-}18)$$

$$f_1(x) = (1-3h)x^2+(h-1)v(v+1) \qquad (1\text{-}19)$$

$$f_2(x) = (2h-3)x^2+(6h-3)x+(h+1)(2a^2+v^2+v) \qquad (1\text{-}20)$$

$$f_3(x) = 2(h+1)x^2+8(1-h)x+4a^2(h+1)+(1-h)v(v+1) \qquad (1\text{-}21)$$

$$f_4(x) = (2-3h)(x^2-x)-4x+2a^2(h+1)-v(v+1) \qquad (1\text{-}22)$$

$$f_5(x) = (h-3)x^2 \qquad (1\text{-}23)$$

$$f_6(x) = x^2+x \qquad (1\text{-}24)$$

Indicial equation: $f_o(\sigma) = 0$ with roots:

$$\sigma_1 = v+1 \quad , \quad \sigma_2 = -v \quad . \qquad (1\text{-}25)$$

Recurrence formula:

$$\sum_{m=0}^{6} e_{n-m} f_m(n+\sigma-m) = 0 \quad ; \quad e_0 = 1 \quad ; \quad e_{-j} = 0 \; , \; j = 1,2,\ldots \qquad (1\text{-}26)$$

For $\sigma = \sigma_1 = v+1$ we always obtain a solution:

$$R(t) = Kt^{\sigma}[1+\sum_{n=1}^{\infty} e_n t^n] \quad , \qquad (1\text{-}27)$$

where K is a constant and the coefficients e_n are determined using (1-26) with $\sigma = v+1$. If $\sigma_1 - \sigma_2 = 2v+1$ is not an integer, use of (1-26) and (1-27) with $\sigma = -v$ yields a second independent solution. If $2v+1$ is equal to a positive integer, the second solution is logarithmic and will be found later.

For n=1 equation (1-26) yields: $e_1 = -f_1(\sigma)/f_0(\sigma+1)$; since in any case: $\sigma(\sigma-1) = v(v+1)$, we obtain:

$$e_1 = \sigma + \frac{1}{2} - \frac{1}{2h} \quad . \qquad (1\text{-}28)$$

As $x \to 0$ and $t \to 0$, we have:

$$t = \frac{x}{x+2a} = \frac{x}{2a}\,\frac{1}{1+(x/2a)} = \frac{x}{2a}[1-\frac{x}{2a}+(\frac{x}{2a})^2+\ldots.]$$

Substituting in (1-27) we obtain:

$$R(t)=K(\frac{x}{2a})^{\sigma}(1+\frac{x}{2a})^{-\sigma}[1+(\sigma+\frac{1}{2}-\frac{1}{2h})\frac{x}{2a}(1+O_{(x)})+O_{(x^2)}] =$$

$$= \frac{K}{(2a)^{\sigma}}x^{\sigma}[1-\sigma\frac{x}{2a}+O_{(x^2)}][1+(\sigma+\frac{1}{2}-\frac{1}{2h})\frac{x}{2a}+O_{(x^2)}] =$$

$$= \frac{K}{(2a)^{\sigma}}x^{\sigma}[1+\frac{h-1}{4ah}x+O_{(x^2)}] = \frac{K}{(2a)^{\sigma}}x^{\sigma}[1-\frac{c}{2ab}x+O_{(x^2)}] \; .$$

Taking $K = (2a)^{\sigma}$ and referring to (1-10),(1-11) and the remarks following them, we see that the two solutions defined by (1-27), (1-26) can be identified with $R_1(x)$ and $R_2(x)$ of (1-10), (1-11) in the corresponding cases $\sigma_1 = v+1$ and $\sigma_2 = -v$. That is:

$$R_{\substack{1\\2}}(x) = x^{\sigma}[1+\sum_{n=1}^{\infty} a_n x^n] = (2a)^{\sigma} t^{\sigma}[1+\sum_{n=1}^{\infty} e_n t^n] \quad . \tag{1-29}$$

Solution R_1 is given for $\sigma=v+1$, R_2 for $\sigma=-v$, if $2v+1$ is not an integer. If $2v+1$ is equal to an integer, R_2 will be found later. The series in x converge uniformly and absolutely for $|x|<\min(|a|,|b|)$, those in t for $|t|<\min(1,|h|)$, providing the analytic continuation of $R_1(x)$, $R_2(x)$ into the right half-plane shown in figure (1-1).

2v+1 IS EQUAL TO AN INTEGER

In this case the second independent solution $R_2(x)$ of equation (I 1-50) becomes logarithmic. The preceding analysis is valid, without alteration, to the solution $R_1(x)$ with $\sigma=v+1$. In the biconical antenna of PART I, as well as in other problems, the second solution $R_2(x)$ appears indirectly. What is actually directly involved is $R_4(x)$, the solution which satisfies the radiation condition at infinity, corresponding to an integral value of v. Evaluation of $R_4(x)$ for small x, where its asymptotic series representation can not be used, involves both $R_1(x)$ and $R_2(x)$ of integral order. An equation like (1-54), PART I, must be available to provide the analytic continuation of $R_4(x)$ in the vicinity of x=0. Thus, not only $R_2(x)$, but its complete asymptotic expansion for large $|x|$, equation (1-53), PART I, must be known.

Following Whittaker and Watson (16 pp. 200-201), we put:

$$R_2(x) = \ln x \; R_1(x) + S(x) \quad , \tag{1-30}$$

where $R_1(x) = \sum_{m=0}^{\infty} a_m x^{m+v+1}$, $a_0 = 1$, is the first solution obtained in the preceding section, and:

$$S(x) = \sum_{n=o}^{\infty} b_n x^{n-v} \quad . \tag{1-31}$$

We write equation (I 1-50) in the following form:

$$\tau_o(x)x^2R''(x)+\tau_1(x)xR'(x)+\tau_2(x)R(x) = 0 \quad , \tag{1-32}$$

where:

$$\tau_o(x) = x^2+(a+b)x+ab \tag{1-33}$$

$$\tau_1(x) = cx \tag{1-34}$$

$$\tau_2(x) = x^4+2ax^3+[a^2-v(v+1)]x^2-v(v+1)(a+b)x-v(v+1)ab \quad . \tag{1-35}$$

Substituting (1-30) into (1-32) we obtain:

$$\tau_o(x)x^2[R_1''(x)\ln x+2\,\frac{R_1'(x)}{x}-\frac{R_1(x)}{x^2}+S''(x)]+\tau_1(x)x[R_1'(x)\ln x+$$

$$+\frac{R_1(x)}{x}+S'(x)]+\tau_2(x)[R_1(x)\ln x+S(x)] = 0 \quad ,$$

or, since $R_1(x)$ satisfies (1-32):

$$\tau_o(x)x^2S''(x)+\tau_1(x)xS'(x)+\tau_2(x)S(x) =$$

$$= \tau_o(x)[R_1(x)-2xR_1'(x)]-\tau_1(x)R_1(x) \quad . \tag{1-36}$$

Substituting (1-31),(1-33)-(1-35) and $R_1(x)=\sum_{m=o}^{\infty} a_m x^{m+v+1}$ in the above equation, we obtain:

$$\sum_{n=o}^{\infty} b_n x^{n-v} f(x,n-v) = \sum_{m=o}^{\infty} a_m x^{m+v+1} F(x,m+v+1) \quad , \tag{1-37}$$

where:

$$f(x,n-v) = (n-v)(n-v-1)[x^2+(a+b)x+ab]+(n-v)cx+x^4+2ax^3+[a^2-$$

$$-v(v+1)]x^2-v(v+1)(a+b)x-v(v+1)ab = \sum_{m=o}^{4} f_m(n-v)x^m$$

and:

$$F(x,m+v+1) = -[2(m+v+1)-1][x^2+(a+b)x+ab]-cx = \sum_{j=0}^{2} F_j(m+v+1)x^j \quad .$$

The following definitions were used:

$$f_0(n-v) = ab[(n-v)(n-v-1)-v(v+1)] = abn(n-2v-1) \tag{1-38}$$

$$f_1(n-v) = (a+b)(n-v)(n-v-1)+(n-v)c-(a+b)v(v+1) \tag{1-39}$$

$$f_2(n-v) = (n-v)(n-v-1)+a^2-v(v+1) \tag{1-40}$$

$$f_3(n-v) = 2a \tag{1-41}$$

$$f_4(n-v) = 1 \tag{1-42}$$

$$F_0(m+v+1) = -ab[2(m+v+1)-1] \tag{1-43}$$

$$F_1(m+v+1) = -(a+b)[2(m+v+1)-1]-c = -2(a+b)(m+v+1)+2b \tag{1-44}$$

$$F_2(m+v+1) = -2(m+v+1)+1 \quad . \tag{1-45}$$

Equation (1-37) can be written:

$$\sum_{n=0}^{\infty} b_n x^n f(x,n-v) = x^{2v+1} \sum_{m=0}^{\infty} a_m x^m F(x,m+v+1)$$

and shows, incindentally, that, with $f(x,n-v)$ and $F(x,m+v+1)$ being polynomials in x, it can not be satisfied unless $2v+1$ is equal to a positive integer. In its right-hand side put:

$$m+2v+1 = n \quad . \tag{1-46}$$

It becomes:

$$\sum_{n=0}^{\infty} b_n x^n f(x,n-v) = \sum_{n=2v+1}^{\infty} a_{n-2v-1} x^n F(x,n-v) \quad .$$

Since $a_{-1} = a_{-2} = \ldots\ldots = 0$, we can also write it as follows:

$$\sum_{n=o}^{\infty} x^n b_n f(x,n-v) = \sum_{n=o}^{\infty} a_{n-2v-1} x^n F(x,n-v) \quad . \tag{1-47}$$

The lowest exponent of x is n=0 and its coefficient is $b_o f_o(0-v)= =0$. Generally for $1 \leqslant n \leqslant 2v$ we have:

$$b_n f_o(n-v)+b_{n-1}f_1(n-v-1)+b_{n-2}f_2(n-v-2)+b_{n-3}f_3(n-v-3)+$$
$$+b_{n-4}f_4(n-v-4) = 0 \quad ; \quad b_{-m}= 0 \ , \quad m = 1,2,3,\ldots \quad . \tag{1-48}$$

For all these values of n we observe that $f_o(n-v) \neq 0$, permitting to use successively the recurrence formula and obtain b_1, b_2, ..., b_{2v}. For n=2v+1, i.e. for the coefficient of x^{2v+1} in (1-47), we have:

$$b_{2v+1}f_o(v+1)+b_{2v}f_1(v)+b_{2v-1}f_2(v-1)+b_{2v-2}f_3(v-2)+b_{2v-3}f_4(v-3) =$$
$$= a_o F_o(v+1) = -ab(2v+1) \quad . \tag{1-49}$$

Now, however, $f_o(v+1) = 0$, according to (1-38). In order to satisfy (1-49), we choose conveniently the value of b_o, left undetermined so far. Notice that b_1, b_2,..., b_{2v}, determined so far with the use of (1-48), are all proportional to b_o. In fact, we can write:

$$b_n= b_o d_n \quad , \quad d_o= 1 \quad , \quad n = 0,1,2,3,\ldots. \tag{1-50}$$

and use (1-48), with d_n in place of b_n, to determine the numbers d_o, d_1, d_2,...., d_{2v} completely, with initial conditions: $d_o= 1$, $d_{-1}= d_{-2}=\ldots= 0$. Then, (1-49) is satisfied if we take:

$$b_o= -\frac{ab(2v+1)}{d_{2v}f_1(v)+d_{2v-1}f_2(v-1)+d_{2v-2}f_3(v-2)+d_{2v-3}f_4(v-3)} \quad . \tag{1-51}$$

For the b_n's (n = 1,2,....,2v) we then use (1-50).

For $n > 2v+1$ equation (1-47) yields:

$$b_n f_o(n-v)+b_{n-1}f_1(n-v-1)+b_{n-2}f_2(n-v-2)+b_{n-3}f_3(n-v-3)+b_{n-4}f_4(n-v-4)=$$
$$= a_{n-2v-1}F_o(n-v)+a_{n-2v-2}F_1(n-v-1)+a_{n-2v-3}F_2(n-v-2) \quad . \qquad (1-52)$$

Remembering that $a_o= 1$, $a_{-1}=a_{-2}=....= 0$, we see that this equation is valid for all integral values of n, including (1-48) as a special case. The supplementary initial conditions are:

$$b_{-3}= b_{-2}= b_{-1}= 0 \ , \quad b_o \text{ as in (1-51)} \quad . \qquad (1-53)$$

Putting n=0 in (1-52), we get: $b_o f_o(0-v)+b_{-4}f_4(0-v-4) = 0$; since $f_o(0-v) = 0$, according to (1-38), we obtain $b_{-4}= 0$. By repeated applications of (1-52) we also find: $b_{-5}= b_{-6}=.....= 0$. Thus, conditions (1-53) are enough to satisfy all requirements for the coefficients b_n of $S(x)$.

The process so far has left b_{2v+1} undetermined. This does not cause any difficulty. Looking at (1-52) we observe that if it can be satisfied by the set:

$$b_o,\ b_1,\ b_2,\ .....,\ b_{2v+1},\ b_{2v+2},\ .....$$

it can also be satisfied by the set:

$$b_o+ka_{-2v-1}= b_o,\ b_1+ka_{-2v}= b_1,\ ...,\ b_{2v+1}+ka_o,\ b_{2v+2}+ka_1,\ .....,$$

where k is a constant. The reason is that the set:

$$a_{-2v-1}= 0,\ a_{-2v}= 0,\ .....,\ a_o,\ a_1,\ .....$$

satisfies the homogeneous part of (1-52). Actually for:

$$n = 2v+1+m \quad , \quad b_n= b_{2v+1+m}= B_m \qquad (1-54)$$

equation (1-52) becomes:

$$B_m f_o(m+v+1)+B_{m-1}f_1(m+v)+B_{m-2}f_2(m+v-1)+B_{m-3}f_3(m+v-2)+B_{m-4}f_4(m+v-3)=$$
$$= a_m F_o(m+v+1)+a_{m-1}F_1(m+v)+a_{m-2}F_2(m+v-1) \quad . \qquad (1-55)$$

Its homogeneous part is satisfied by the set a_o, a_1, ... , as a comparison of (1-38)-(1-42), for $n=m+2v+1$ (or $n-v=m+v+1$), with (1-1)-(1-5), for $n+\sigma=m+v+1$, reveals; equations (1-1)-(1-5) with $n+\sigma=m+v+1$ give the coefficients of the recurrence relation (1-8) for the coefficients a_m of $R_1(x)$.

We can choose $b_{2v+1}= B_o$ arbitrarily. The simplest choice is:

$$b_{2v+1} = B_o = 0 \ , \tag{1-56}$$

in the process of using (1-52) for the b_n's. A non-vanishing value for $b_{2v+1}= B_o$ means simply, that $S(x)$, and consequently $R_2(x)$, contain the additive solution $b_{2v+1}\, R_1(x)$, which can be discarded in the definition of the second independent solution $R_2(x)$. To be definite then, we define $R_2(x)$ as follows:

$$R_2(x) = \ln x\, R_1(x)+x^{-v}\sum_{n=o}^{\infty} b_n x^n \ ; \ |x|<\min(|a|,|b|) \ ;$$
$$b_{2v+1}= 0 \ , \ b_o \ \text{ as in (1-51)} \ . \tag{1-57}$$

For the rest of the b_n's we use (1-52),(1-53). This definition leaves no ambiguity as to which particular $R_2(x)$ we consider.

Analytic Continuation of $R_2(x)$ in the Right-Half x-Plane:

We use the same transformation (1-15), as for $R_1(x)$. In analogy with (1-30) we write:

$$R_2(t) = \ln \frac{2at}{1-t}\, R_1(t)+S(t) \quad , \quad S(t) = \sum_{n=o}^{\infty} c_n t^{n-v} \ . \tag{1-58}$$

For a moment, designating the coefficients of (1-16) by $T_o(t)$, $T_1(t)$, $T_2(t)$, we can write this equation as follows:

$$T_o(t)R''(t)+T_1(t)R'(t)+T_2(t)R(t) = 0 \ . \tag{1-59}$$

Substitute (1-58) in the above equation:

$$T_o(t)[S''+\ln(\tfrac{2at}{1-t})R_1''+\frac{2}{t(1-t)}R_1'-\frac{1-2t}{t^2(1-t)^2}R_1]+T_1(t)[S'+\ln(\tfrac{2at}{1-t})R_1'+$$

$$+\frac{R_1}{t(1-t)}]+T_2(t)[\ln(\tfrac{2at}{1-t})R_1+S] = 0 \quad .$$

Since $R_1(t)$ itself satisfies (1-59), we obtain:

$$T_o(t)S''(t)+T_1(t)S'(t)+T_2(t)S(t) = [\frac{1-2t}{t^2(1-t)^2}T_o(t)-\frac{T_1(t)}{t(1-t)}]R_1(t)-$$

$$-\frac{2T_o(t)}{t(1-t)}R_1'(t) \quad .$$

Writing $T_o(t)$, $T_1(t)$, $T_2(t)$ in full as in (1-16) and using:

$$S(t) = \sum_{n=o}^{\infty} c_n t^{n-v} \quad , \quad R_1(t) = (2a)^{v+1}t^{v+1}[1+\sum_{m=1}^{\infty} e_m t^m]$$

we obtain:

$$\sum_{n=o}^{\infty} t^{n-v}c_n f(t,n-v)=(2a)^{v+1}\sum_{m=o}^{\infty} e_m t^{m+v+1}[(1-2t)(t+h)(t-1)^2(t+1)+$$

$$+t(t-1)^3(1-h)+2t(t-1)^2(t+1)(t+h)+(m+v+1)2(t-1)^2(t^2-1)(t+h)] =$$

$$= (2a)^{v+1}\sum_{m=o}^{\infty} e_m t^{m+v+1} F(t,m+v+1) \quad , \tag{1-60}$$

where:

$$F(t,m+v+1) = (m+v+1)[2t^5+(2h-4)t^4-4ht^3+4t^2+(4h-2)t-2h]+$$

$$+[(2-h)t^4+4(h-1)t^3+(2-4h)t^2+h] = \sum_{q=o}^{5} F_q(m+v+1)t^q \tag{1-61}$$

$$f(t,n-v) = \sum_{q=o}^{6} f_q(n-v)t^q \quad . \tag{1-62}$$

The functions appearing in the last equation are exactly the ones

that appeared previously in $R_1(t)$; thus, $f_o(x)$ to $f_6(x)$ are defined by equations (1-18) to (1-24). On the other hand, the following new functions were introduced in (1-61):

$$F_o(x) = -h(2x-1) \tag{1-63}$$

$$F_1(x) = (4h-2)x \tag{1-64}$$

$$F_2(x) = 4x+2-4h \tag{1-65}$$

$$F_3(x) = -4hx+4(h-1) \tag{1-66}$$

$$F_4(x) = (h-2)(2x-1) \tag{1-67}$$

$$F_5(x) = 2x \quad . \tag{1-68}$$

As with $R_2(x)$, we call:

$$c_n = H_o(2a)^{v+1}\, g_n \quad ; \quad g_o = 1 \quad ; \quad g_{-m}=0 \ , \ m = 1,2,\ldots \tag{1-69}$$

and write (1-60) in the following form:

$$H_o\sum_{n=o}^{\infty} t^n g_n f(t,n-v) = t^{2v+1}\sum_{m=o}^{\infty} e_m t^m\, F(t,m+v+1) \quad . \tag{1-70}$$

For $1 \leqq n \leqq 2v$ this equation yields the following recurrence formula:

$$\sum_{q=o}^{6} g_{n-q} f_q(n-v-q) = 0 \quad ; \quad g_o = 1 \quad ; \quad g_{-m} = 0 \quad , \ m = 1,2,.. \tag{1-71}$$

with the use of which g_1, g_2, ..., g_{2v} are evaluated. For n=1 it yields:

$$g_1 = -f_1(-v)/f_o(1-v) = -v+\frac{1}{2}-\frac{1}{2h} \quad . \tag{1-72}$$

For n=2v+1 equation (1-70) yields:

$$H_o[g_{2v+1}f_o(v+1)+g_{2v}f_1(v)+g_{2v-1}f_2(v-1)+\ldots+g_{2v-5}f_6(v-5)]=F_o(v+1)=$$

$= -h(2v+1)$.

Since $f_o(v+1) = 0$, according to (1-18), this equation serves simply to define H_o, i.e.

$$H_o = -\frac{h(2v+1)}{g_{2v}f_1(v)+g_{2v-1}f_2(v-1)+g_{2v-2}f_3(v-2)+g_{2v-3}f_4(v-3)+g_{2v-4}f_5(v-4)+g_{2v-5}f_6(v-5)} \quad . \tag{1-73}$$

For $n > 2v+1$ equation (1-70) yields the following recurrence formula for the remaining coefficients g_n:

$$\sum_{q=o}^{6} g_{n-q} f_q(n-v-q) = \frac{1}{H_o} \sum_{q=o}^{5} e_{n-2v-1-q} \, F_q(n-v-q) \quad . \tag{1-74}$$

As before, the process has left g_{2v+1} undetermined. It will be chosen in such a way as to identify the present solution $R_2(t)$ with the solution $R_2(x)$ defined explicitly in (1-57). Comparison of (1-58) with (1-30),(1-31) and (1-15) expresses this requirement as follows:

$$S(t) = H_o(2a)^{v+1} t^{-v} \sum_{n=o}^{\infty} g_n t^n = S(x) = x^{-v} b_o \sum_{n=o}^{\infty} d_n x^n \quad , \tag{1-75}$$

where as in (1-50) we can write $b_n = b_o d_n$ for all n. Substituting $t = \frac{x}{x+2a} = \frac{x}{2a}(1+\frac{x}{2a})^{-1}$ we obtain:

$$H_o \sum_{n=o}^{\infty} g_n (\frac{x}{2a})^n (1+\frac{x}{2a})^{v-n} = \frac{b_o}{(2a)^{2v+1}} \sum_{n=o}^{\infty} d_n x^n \quad . \tag{1-76}$$

For sufficiently small $|x|$ the binomial theorem yields:

$$(1+\frac{x}{2a})^{v-n} = 1+(v-n)\frac{x}{2a} + \ldots\ldots.$$

$$+\frac{(v-n)(v-n-1)\ldots\ldots(v-n-m+1)}{m!}\left(\frac{x}{2a}\right)^m+\ldots\ldots\ldots \quad . \tag{1-77}$$

If v-n is equal to a positive integer, it reduces to a finite polynomial. Assuming v itself to be an integer (if v=n+1/2 the procedure remains the same), we write (1-76) as follows:

$$\begin{aligned}
H_o\Big\{&1+v\,\frac{x}{2a}+\frac{v(v-1)}{2!}\left(\frac{x}{2a}\right)^2+\ldots\ldots+\left(\frac{x}{2a}\right)^v+\\
&+g_1\left(\frac{x}{2a}\right)\left[1+(v-1)\,\frac{x}{2a}+\ldots\ldots+\left(\frac{x}{2a}\right)^{v-1}\right]+\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+g_v\left(\frac{x}{2a}\right)^v+\\
&+g_{v+1}\left(\frac{x}{2a}\right)^{v+1}\left[1-\frac{x}{2a}+\left(\frac{x}{2a}\right)^2-\left(\frac{x}{2a}\right)^3+\ldots\ldots\right]+\\
&+g_{v+2}\left(\frac{x}{2a}\right)^{v+2}\left[1-2\,\frac{x}{2a}+\frac{2\cdot 3}{2!}\left(\frac{x}{2a}\right)^2+\ldots\right]+\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+g_{2v}\left(\frac{x}{2a}\right)^{2v}\left[1-v\,\frac{x}{2a}+\frac{v(v+1)}{2!}\left(\frac{x}{2a}\right)^2+\ldots\ldots\right]+\\
&+g_{2v+1}\left(\frac{x}{2a}\right)^{2v+1}\left[1-(v+1)\,\frac{x}{2a}+\frac{(v+1)(v+2)}{2!}\left(\frac{x}{2a}\right)^2+\ldots\ldots\right]+\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big\}=\\
&=\frac{b_o}{(2a)^{2v+1}}\left[1+d_1x+\ldots\ldots+0x^{2v+1}+d_{2v+2}x^{2v+2}+\ldots\ldots\right]
\end{aligned}$$

We immediately deduce that:

$$H_o=\frac{b_o}{(2a)^{2v+1}} \tag{1-78}$$

and from the coefficient of x^{2v+1}:

$$g_{2v+1} = (-1)^{v+1}g_{v+1}+(-1)^{v+2}vg_{v+2}+(-1)^{v+3}\frac{v(v-1)}{2}g_{v+3}+\dots\dots$$
$$\dots+(-1)^{v+m}\frac{v(v-1)\dots(v-m+2)}{(m-1)!}g_{v+m}+\dots+vg_{2v} \quad . \qquad (1-79)$$

For $v=n+1/2$ this formula is modified by inclusion of some additional terms. With this definition (1-74) can be used to yield all g_n's for $n>2v+1$. The second solution, defined precisely in (1-57), can also be written:

$$R_2(x)=R_2(t)=\ln x\ R_1(x)+\frac{H_o(2a)^{v+1}}{t^v}[1+\sum_{n=1}^{\infty}g_n t^n];\ |t|<\min(1,|h|)\ (1-80)$$

and so, in connection with (1-29), can be analytically continued into the right-half x-plane shown in figure (1-1).

Equation (1-78) can be checked easily for low order functions (for example $v=1,2,3$), using directly the definitions (1-51), (1-73). For large v the numerical computations of Chapter 3, PART I, checked all relations and proved them correct. In Cases I and II, both series in x and t were used to evaluate $R_1(x)$, $R_2(x)$; the results were identical. It was also found that, while the series in x for both $R_1(x)$ and $R_2(x)$ had the same rate of convergence, the series for $R_1(t)$ was better and could be used farther than the series for $R_2(t)$. The coefficients g_n increase faster than the e_n's. As the order v of the functions increases, both e_n and g_n increase faster, rendering the rate of convergence of both series in t poorer. In any case, the series in t could be used for larger $|x|$ than the series in x and were able to carry the computations into the region of validity of the asymptotic series (especially for x close to the real axis).

ASYMPTOTIC SOLUTIONS $R_3(x)$, $R_4(x)$ AROUND $x = \infty$

As $x \to \infty$ the coefficients of $R'(x)$ and $R''(x)$ in equation (I 1-50) vary as:

$$\frac{c}{(x+a)(x+b)} \sim O(x^{-2}) \quad , \qquad 1+\frac{c}{x+b} - \frac{v(v+1)}{x^2} \sim 1+O(x^{-1}) \quad .$$

Point $x=\infty$ is an irregular singularity of rank 1 (10 pp. 58-77). We follow the method of Chapter III in reference 10. First, we eliminate $R'(x)$ by putting:

$$R(x) = \sqrt{\frac{x+a}{x+b}}\, v(x) = y(x)v(x) \quad . \tag{1-81}$$

Points $x=-a$ and $x=-b$ are branch points of the general solution of equation (I 1-50) and in the cut x-plane with branch lines from $x=-a$ and $x=-b$ to ∞ (along the negative real axis preferably), the functions $R(x)$, $y(x)=\sqrt{(x+a)/(x+b)}$, $v(x)$ in (1-81) are analytic and single-valued. We obtain for $v(x)$:

$$v''(x)+\left[2(y'/y)+\frac{c}{(x+a)(x+b)}\right]v'(x)+\left[(y'/y)' +(y'/y)^2+(y'/y)\frac{c}{(x+a)(x+b)}+\right.$$

$$\left.+1+\frac{c}{x+b} - \frac{v(v+1)}{x^2}\right]v(x) = 0 \; .$$

But $y'/y = -c/[2(x+a)(x+b)]$. So, we get:

$$v''(x) + q(x)v(x) = 0 \quad , \tag{1-82}$$

where:

$$q(x) = 1+\frac{c}{x+b} - \frac{v(v+1)}{x^2} + \frac{1/4}{(x+b)^2} - \frac{3/4}{(x+a)^2} + \frac{1/2}{(x+a)(x+b)} = 1+$$

$$+\frac{c}{x}\left(1+\frac{b}{x}\right)^{-1} - \frac{v(v+1)}{x^2} + \frac{1}{4x^2}\left(1+\frac{b}{x}\right)^{-2} - \frac{3}{4x^2}\left(1+\frac{a}{x}\right)^{-2} + \frac{1}{2cx}\left[\left(1+\frac{b}{x}\right)^{-1} -\right.$$

$-(1+\frac{a}{x})^{-1}]$.

For $|x| > \max(|a|, |b|)$ we can use the binomial theorem to obtain the expansion:

$$q(x) = \sum_{n=o}^{\infty} q_n x^{-n} \quad . \tag{1-84}$$

A few manipulations yield:

$$q_o = 1 \tag{1-85}$$

$$q_1 = c \tag{1-86}$$

$$q_2 = -v(v+1)-cb \tag{1-87}$$

$$q_3 = c(b^2+1) \tag{1-88}$$

....

$$q_n = (-1)^n[b^{n-1}(\frac{n-1}{4b} - \frac{1}{2c} - c) + a^{n-1}(\frac{1}{2c} - \frac{3(n-1)}{4a})], \; n=3,4,\ldots \tag{1-89}$$

We now try to satisfy (1-82) with the formal series:

$$v(x) = e^{\omega x} \sum_{n=o}^{\infty} h_n x^{-\rho-n}, \; h_o \neq 0 \; , \; |x| > \max(|a|, |b|) \quad . \tag{1-90}$$

Assuming $q_{-m} = h_{-m} = 0$ ($m=1,2,3,\ldots.$) and substituting (1-90) and (1-84) into (1-82) we obtain, after cancelling the factor $e^{\omega x}$ (10 pp. 58-77):

$$\omega^2 \sum_{n=-\infty}^{\infty} h_n x^{-\rho-n} - 2\omega \sum_{n=-\infty}^{\infty} (n+\rho) h_n x^{-\rho-n-1} + \sum_{n=-\infty}^{\infty} (n+\rho)(n+\rho+1) h_n x^{-\rho-n-2} +$$

$$+ \sum_{m=-\infty}^{\infty} q_m x^{-m} \sum_{k=-\infty}^{\infty} h_k x^{-\rho-k} = 0 \quad .$$

Putting the coefficient of $x^{-\rho-n}$ equal to zero we obtain:

$$\omega^2 h_n - 2\omega(n+\rho-1)h_{n-1} + (n+\rho-2)(n+\rho-1)h_{n-2} + \sum_{m=o}^{n} q_m h_{n-m} = 0 \qquad (1\text{-}91)$$

for $n=0,1,2,\ldots.$ (for $n=-1,-2,\ldots.$ it is automatically satisfied, since $q_{-m} = h_{-m} = 0$, $m=1,2,\ldots.$).

For $n=0$ we get: $\omega^2+q_o = 0$ or $\omega^2+1 = 0$. That is $\omega_1 = i$, $\omega_2 = -i$. For $n=1$ and since $\omega^2+1 = 0$, we obtain:

$-2\omega\rho+q_1 = 0$ or $\rho = c/2\omega$, i.e. $\rho_1 = -ic/2$, $\rho_2 = ic/2$.

From (1-91), replacing n by n+1 and using $\omega^2+1 = 0$, $\rho = c/2\omega$, we also obtain the following recurrence formula for the h_n's:

$$2\omega n h_n = (n+c/2\omega)(n+c/2\omega-1)h_{n-1} + \sum_{m=2}^{n+1} q_m h_{n+1-m}, \quad n=1,2,\ldots. \qquad (1\text{-}92)$$

Taking $h_o = 1$ we can find:

$$h_1 = \frac{(c/2\omega)(c/2\omega+1) - v(v+1) - cb}{2\omega}$$

and so on. We then obtain two particular normal solutions around $x=\infty$:

$$R_{\substack{3\\4}}(x) \sim \sqrt{\frac{x+a}{x+b}}\, e^{\omega x} x^{-\rho}\Big[1+\sum_{n=1}^{\infty} h_n x^{-n}\Big], \quad |x| > \max(|a|,|b|), \qquad (1\text{-}93)$$

where $R_3(x)$ is given for $\omega_1 = i$, $\rho_1 = -ic/2$ and $R_4(x)$ for $\omega_2 = -i$, $\rho_2 = ic/2$. The series appearing in the above formal representations are normal asymptotic series in the precise sense of Poincaré's definition (9 pp. 168-174 444-445, 10 pp. 69-72), in the region $|x|>\max(|a|,|b|)$. Expanding according to the binomial:

$$\sqrt{\frac{x+a}{x+b}} = \Big(1+\frac{a}{x}\Big)^{1/2}\Big(1+\frac{b}{x}\Big)^{-1/2} = (1+a/2x+\ldots)(1-b/2x+\ldots.) =$$

$$= 1+c/2x+\ldots., \quad |x|>\max(|a|,|b|)$$

we obtain the alternative expressions:

$$R_{\frac{3}{4}}(x) \sim e^{\omega x}x^{-\rho}[1+ \frac{(c/2\omega)(c/2\omega+1)-v(v+1)-cb}{2\omega}\frac{1}{x}+\ldots][1+c/2x+\ldots.]=$$

$$= e^{\omega x}x^{-\rho}[1+(\frac{c}{2}+\frac{(c/2\omega)(c/2\omega+1)-v(v+1)-cb}{2\omega})\frac{1}{x}+\frac{g_2}{x^2}+$$

$$+\frac{g_3}{x^3}+\ldots\ldots] , \tag{1-94}$$

which will serve for comparison later. Another useful remark is the following: in the real case (i.e. a,b real) we have $\rho_1=\bar{\rho}_2=-ic/2$, where the bar signifies the complex conjugate, while always $\omega_1=\bar{\omega}_2=i$. Then, according to (1-85)-(1-89), all q_n's are real and the recurrence formula (1-92) yields as coefficients h_n for $R_3(x)$ the complex conjugates of the coefficients of $R_4(x)$. The same is obviously true for the coefficients g_n in (1-94). This means:

$$R_3(x) = \overline{R_4(\bar{x})} \quad , \quad a,b \ \text{real} , \tag{1-95}$$

or for real x:

$$R_3(x) = \overline{R_4(x)} \quad . \tag{1-96}$$

The critical line, or Stokes line, is given by (10 p. 72): $Re(\omega x) = 0$. If $x=x_r+ix_i$, with $\omega = \pm i$ we have:

$$Re[\pm i(x_r+ix_i)] = \pm x_i = 0 \quad \text{or} \quad x_i = 0 \ .$$

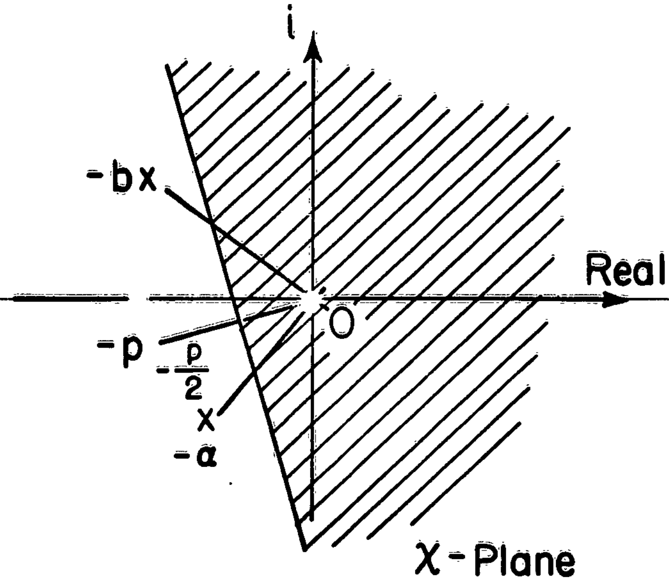
So, the real x-axis is the critical or Stokes line. Any solution R(x) of equation (I 1-50) can be expressed asymptotically as a linear combination of the above formal solutions R_3, R_4, i.e.

$$R(x) \underset{x\to\infty}{\simeq} A_3\, R_3(x)+A_4\, R_4(x) \tag{1-97}$$

and this expression holds uniformly in each of the upper and lower half x-planes separated by the Stokes line $x_i= 0$. The

coefficients A_3 and A_4 may, of course, change values from one plane to another for the same solution $R(x)$ (10 p. 73). A similar combination expresses $R(x)$ along the real axis.

The main problem of this research is to find explicitly expressions like (1-97) for the particular solutions $R_1(x)$, $R_2(x)$ defined in the preceding section, in both halves of the x-plane and along the real axis. This is the subject of Chapter 2. One last remark concerning the evaluation of $R_3(x)$ and $R_4(x)$. For complex a, b, x it was observed, that the asymptotic series for $R_4(x)$ could be used earlier, for smaller $|x|$, than the asymptotic series for $R_3(x)$. The order ν did not affect much the value $|x|$, after which the asymptotic series could be used.



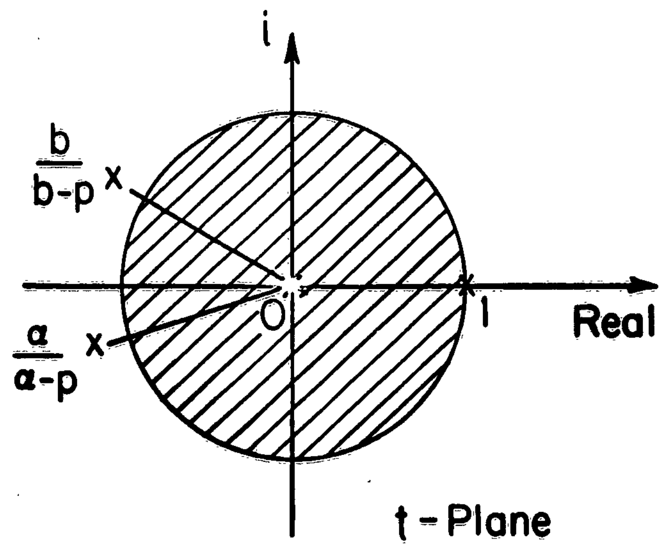


Fig. 1-1 Comformal mapping of χ-plane on to t-plane according to (1-13).

CHAPTER 2

ASYMPTOTIC EXPANSIONS OF $R_1(z)$, $R_2(z)$ FOR LARGE $|z|$

INTRODUCTION

We are going to employ the method of W. B. Ford as developed and applied in Chapter VIII of his book, reference 7. Starting with $R_1(z)$, $R_2(z)$ as defined solely by:

$$R_1(z)=z^{v+1}[1+\sum_{n=1}^{\infty} a_n z^n] \ , \quad |z| < \min(|a|,|b|) \qquad (1\text{-}10)$$

$$R_2(z)=z^{-v}[1+\sum_{n=1}^{\infty} b_n z^n] \ , \quad |z|<\min(|a|,|b|) \ , \quad 2v+1\neq\text{integer} \qquad (1\text{-}11)$$

$$a_n f_0(n+\sigma)+a_{n-1}f_1(n+\sigma-1)+a_{n-2}f_2(n+\sigma-2)+a_{n-3}f_3(n+\sigma-3)+$$
$$+a_{n-4}f_4(n+\sigma-4) = 0 \ ; \quad a_{-m}= 0 \ , \ m = 1,2,3,.... \qquad (1\text{-}8)$$

around $z=0$, that is without reference to the fact that they are solutions of:

$$R''(z)+ \frac{c}{(z+a)(z+b)} R'(z)+[1+ \frac{c}{z+b} - \frac{v(v+1)}{z^2}]R(z) = 0 \ , \qquad (\text{I } 1\text{-}50)$$

we are going to obtain their asymptotic expansions. More precisely, in a manner independent of the previous results, we will arrive for $R_1(z)$, and when $2v+1$ is not an integer, for $R_2(z)$ to expressions like:

$$R(z) \underset{z\to\infty}{\sim} A_3 \ R_3(z) + A_4 \ R_4(z) \qquad (1\text{-}97)$$

and at the same time determine explicitly the values of the constants A_3, A_4. The procedure will also yield the same functions

$R_3(z)$ and $R_4(z)$ in exact agreement with:

$$R_{\substack{3\\4}}(z) \sim e^{\omega z} z^{-\rho}\left[1+\left(\frac{c}{2}+\frac{(c/2\omega)(c/2\omega+1)-v(v+1)-cb}{2\omega}\right)\frac{1}{z}+\frac{g_2}{z^2}+\right.$$

$$\left.+\frac{g_3}{z^3}+\ldots\ldots\right] ; \quad \omega_{\substack{3\\4}} = \pm i \;, \quad \rho_{\substack{3\\4}} = \mp ic/2 \;. \qquad (1\text{-}94)$$

In a subsequent section Ford's method will be extended and applied to yield the required results when $2v+1$ is an integer. Ford does not treat this case in his book but refers to it as a subject for further research. Since odd integral values of v appear in the solution $R_4(z)$ used in region (2), figure (1-2), PART I, outside the antenna, we must find an expression like (1-97) for the logarithmic solution $R_2(z)$. Then we will be able to obtain the linear combination $R_4(z)=A_{41}R_1(z)+A_{42}R_2(z)$, permitting the evaluation of $R_4(z)$ for small $|z|$.

We are going to make use of two fundamental theorems proved in Chapters I and VI of Ford's book and refer to them as Theorems I and VI, respectively. They are as follows:

Theorem I: " If the coefficient g(n) of the power series

$f(z) = \sum_{n=0}^{\infty} g_n z^n$, radius of convergence >0,

may be considered as a function $g(w)$ of the complex variable $w = x+iy$ and as such satisfies the two following conditions when considered throughout any arbitrary right half plane $x > x_o$:

(a) is single-valued and analytic

(b) is such that for all $|y|$ sufficiently large one may write $|g(x+iy)| < Ke^{\epsilon|y|}$, where ϵ is an arbitrary small positive quantity given in advance and where K depends only upon x_o and ϵ, then the function $f(z)$ defined as above, is analytic throughout any sector S (vertex at origin) of the z-plane which does not include the positive half of the real axis and $f(z)$ within S is develo-

pable asymptoticaly as follows:

$$f(z) \sim -\sum_{n=1}^{\infty} \frac{g(-n)}{z^n} \quad . "$$

The theorem is supplemented by some remarks and generalizations; we shall make use of the following:

(i) The theorem is valid if $g(w)$, besides satisfying condition (a), is such that we may write, when $x > x_o$ and $|w|$ is large, $|g(w)| < K|w|^c$, where K and c are constants of which the latter may be positive, negative, or zero.

(ii) If $f(z)$ is defined by a power series of the type

$$f(z) = \sum_{n=o}^{\infty} g(n)(z/\mu)^n,$$

where $\mu=\sigma e^{i\psi}$ is a constant, then the theorem continues as before, provided conditions (a) and (b) are satisfied, except that the excluded ray instead of $\arg z = 0$ is now $\arg z = \psi$.

<u>Theorem VI:</u> " Let $f(z)$ be a function of the complex variable z defined by the series:

$$f(z) = \sum_{n=o}^{\infty} \frac{h(n)z^n}{\Gamma(n+p)} \quad ,$$

in which p is any constant (real or complex) and in which h(n) may be regarded as a function h(w) of the complex variable $w = x+iy$ and as such satisfies the two following conditions:

(a) $\dfrac{h(w)}{\Gamma(w+p)}$ is a single-valued, analytic function of w throughout the finite w-plane,

(b) h(w) is such that, when considered for values of w of large modulus lying in the right half plane $R(w)=x > x_o$, where x_o is some assignable number, it may be expressed in the form:

$$h(w)=c_o + \frac{c_1}{w+p} + \frac{c_2}{(w+p)(w+p+1)} + \ldots + \frac{c_s+\delta(w,s)}{(w+p)(w+p+1)\ldots(w+p+s-1)}$$

in which the c_i are constants and $\lim_{|w|\to\infty} \delta(w,s) = 0$; $s=0,1,2,\ldots$ Then, for values of z of large modulus, the function $f(z)$ has the following asymptotic expansions:

$$f(z) \sim -\sum_{n=1}^{\infty} \frac{h(-n)}{\Gamma(p-n)z^n} \quad ; \quad -\pi/2 < \arg z < 3\pi/2 \ ,$$

$$f(z) \sim e^z z^{1-p} \sum_{n=0}^{\infty} \frac{c_n}{z^n} \quad ; \quad -\pi/2 < \arg z < \pi/2 \ ,$$

in which latter development it is understood that, if $z = \rho e^{i\phi}$, we take:

$$z^{1-p} = e^{[(1-p)(\ln\rho + i\phi)]} \quad ; \quad -\pi < \phi \leqq \pi \ .$$

Moreover, if in (b) the quantity x_0 may be regarded as an arbitrary large negative number, the function $f(z)$ is developable asymptotically when $\arg z = \pm\,\pi/2$ in the form of the sum of the series in the above two expressions under the same interpretation for z^{1-p}."

The following remark can be added: For $-\pi/2 < \arg z < \pi/2$ an equivalent asymptotic expansion is:

$$f(z) \sim -\sum_{n=1}^{\infty} \frac{h(-n)}{\Gamma(p-n)z^n} + e^z z^{1-p} \sum_{n=0}^{\infty} \frac{c_n}{z^n} \ ; \ -\pi/2 < \arg z < \pi/2 \ .$$

The added series asymptotically contributes nothing to the expansion. In fact, by factoring out e^z we have:

$$f(z) \sim e^z\left[-e^{-z} \sum_{n=1}^{\infty} \frac{h(-n)}{\Gamma(p-n)z^n} + z^{1-p} \sum_{n=0}^{\infty} \frac{c_n}{z^n}\right] \ .$$

Now for $-\pi/2 < \arg z < \pi/2$ e^{-z} has the following asymptotic expansion: $e^{-z} \sim 0 + \frac{0}{z} + \frac{0}{z^2} + \ldots$. Then the same is true for

$$e^{-z}\sum_{n=1}^{\infty}\frac{h(-n)}{\Gamma(p-n)z^n} \sim 0+\frac{0}{z}+\frac{0}{z^2}+\ldots\ldots , \quad -\pi/2 < \arg z < \pi/2 .$$

Therefore, the two expansions are asymptotically the same.

SOLUTION OF THE ASSOCIATED DIFFERENCE EQUATION

What essentially defines $R_1(z)$ and $R_2(z)$, as given by (1-10), (1-11), is the recurrence formula (1-8), together with (1-1)-(1-5), for the coefficients a_n. Following Ford (7 Chapt. VIII), we replace the index n by the continuous variable x and a_n by the general function u(x). That is, for x = n, n being an integer, we have:

$$u(n) = a_n \quad . \tag{2-1}$$

Then, the recurrence formula (1-8) transforms into the difference equation:

$$\sum_{m=0}^{4} f_m(x+\sigma-m)u(x-m) = 0 \quad .$$

Advancing x to x+4, i.e. writing x+4 for x, we get:

$$\sum_{m=0}^{4} f_m(x+\sigma+4-m)u(x+4-m) = 0 \quad . \tag{2-2}$$

We next substitute:

$$x+\sigma = y \tag{2-3}$$

$$u(x) = v(x+\sigma) = v(y) \tag{2-4}$$

and the equation becomes:

$$p_4(y)v(y+4)+p_3(y)v(y+3)+p_2(y)v(y+2)+p_1(y)v(y+1)+ \\ +p_o(y)v(y) = 0 , \tag{2-5}$$

where, with the use of (1-1)-(1-5), we have:

$$p_4(y)=f_o(x+\sigma+4)=f_o(y+4)=ab[(y+4)(y+3)-v(v+1)]=$$
$$=ab[(y+4)(y+5)-2(y+4)-v(v+1)] \qquad (2\text{-}6)$$

$$p_3(y)=f_1(x+\sigma+3)=f_1(y+3)=(a+b)(y+3)(y+2)+c(y+3)-(a+b)v(v+1)=$$
$$=(a+b)(y+3)(y+4)-(a+3b)(y+3)-(a+b)v(v+1) \qquad (2\text{-}7)$$

$$p_2(y)=f_2(x+\sigma+2)=f_2(y+2)=(y+2)(y+1)+a^2-v(v+1)=$$
$$=(y+2)(y+3)-2(y+2)+a^2-v(v+1) \qquad (2\text{-}8)$$

$$p_1(y)=f_3(x+\sigma+1)=f_3(y+1)=2a \qquad (2\text{-}9)$$

$$p_o(y)=f_4(x+\sigma)=f_4(y)=1 \qquad (2\text{-}10)$$

The last expressions for the $p(y)$'s have been written down as the suitable forms for the solution of equation (2-5) by the method of Laplace's transformation (7 Chapt. VIII, 17 Chapt. XV pp. 478-501, 18 Chapt. III pp. 57-88).

In order to be able to apply Theorems I and VI we must find that particular analytic solution $\bar{v}(y)$ of (2-5), which satisfies condition (2-1), or, in terms of y and $\bar{\bar{v}}(y)$, the conditions:

$$\bar{\bar{v}}(n+v+1)=a_n \text{ for all integers n and for } R_1(z) \qquad (2\text{-}11)$$

$$\bar{\bar{v}}(n-v)=b_n \text{ for all integers n, 2v+1 not an integer,}$$
$$\text{and for } R_2(z)\ . \qquad (2\text{-}12)$$

Equation (2-5) is a linear difference equation of the fourth order with polynomial coefficients. It possesses four independent solutions. For the general theory of these equations and especially for certain of its results, of which continuous use will be made in this analysis, we refer to Chapters I, III of reference 18, Chapters XII and XV of reference 17 and Chapter VIII of

reference 7.

The method of Laplace's transformation is applied by assuming a solution in the form:

$$v(y) = \frac{1}{2\pi i}\int_{\ell} t^{y-1}\,\Psi(t)\,dt\ , \tag{2-13}$$

where ℓ is a line of integration in the complex t-plane, suitably determined later and $\Psi(t)$ is found from a certain differential equation, as will be seen in the following. First, we note that by integration by parts we can get:

$$(y+s)\int_{\ell} t^{y+s-1}\,\Psi dt = \int_{\ell}\frac{\partial t^{y+s}}{\partial t}\,\Psi dt = [t^{y+s}\Psi(t)]_{\ell} - \\ - \int_{\ell} t^{y+s}\Psi'(t)dt \tag{2-14}$$

$$(y+s)(y+s+1)\int_{\ell} t^{y+s-1}\Psi dt = (y+s+1)[t^{y+s}\Psi(t)]_{\ell} - \\ -[t^{y+s+1}\Psi'(t)]_{\ell} + \int_{\ell} t^{y+s+1}\Psi''(t)dt\ . \tag{2-15}$$

Substituting (2-13) into (2-5) and using (2-14),(2-15) and the last expressions for the p(y)'s in (2-6)-(2-10), we easily obtain:

$$\int_{\ell} t^{y-1}[t^2\phi_2(t)\Psi''(t) - t\phi_1(t)\Psi'(t) + \phi_0(t)\Psi(t)]dt + [I(\Psi,t)]_{\ell} = 0\ ,$$

where:

$$\phi_2(t) = abt^4 + (a+b)t^3 + t^2 = abt^2(t+1/a)(t+1/b) \tag{2-16}$$

$$\phi_1(t) = -2abt^4 - (a+3b)t^3 - 2t^2 = -2t^2[abt^2 + (a+b)t + 1] + ct^3 \tag{2-17}$$

$$\phi_0(t) = -abv(v+1)t^4 - (a+b)v(v+1)t^3 + [a^2 - v(v+1)]t^2 + 2at + 1 = \\ = -v(v+1)abt^2(t+1/a)(t+1/b) + a^2(t+1/a)^2\ . \tag{2-18}$$

It can now be seen how the last expressions for the p(y)'s in (2-6)-(2-10) are suitable for the formation of the $\phi(t)$'s.

We also have for $I(\Psi,t)$ (17 pp. 478-501, 18 pp. 57-88):

$$I(\Psi,t) = \Psi(t)[t^y\phi_1(t)+\frac{d}{dt}(t^{y+1}\phi_2(t)]-\Psi'(t)t^{y+1}\phi_2(t) =$$

$$= \Psi(t)t^y[\phi_1(t)+(y+1)\phi_2(t)+t\phi_2'(t)]-\Psi'(t)t^{y+1}\phi_2(t) \quad . \qquad (2\text{-}19)$$

Since $\phi_1(t)+t\phi_2'(t) = -2abt^2(t+1/a)(t+1/b)+ct^3+4abt^4+3(a+b)t^3+2t^2 =$ $= 2abt^3(t+1/b)$, we also have:

$$I(\Psi,t) = abt^{y+2}(t+1/b)\left\{\Psi(t)[(y+1)(t+1/a)+2t]-\Psi'(t)t(t+1/a)\right\} \quad (2\text{-}20)$$

We conclude that (2-13) provides a solution of (2-5) if $\Psi(t)$ is a solution of the differential equation:

$$abt^4(t+1/a)(t+1/b)\Psi''(t)-t\phi_1(t)\Psi'(t)+\phi_0(t)\Psi(t) = 0 \qquad (2\text{-}21)$$

and the path of integration ℓ is chosen so that $I(\Psi,t)$ has the same value at both extremities of the path when it is open, or so that $I(\Psi,t)$ returns to the same value if ℓ is closed and t returns to the same point after describing it.

We now look for the behaviour of the solutions of (2-21) at the vicinity of its singularities. The equation has three regular singularities at $t_1=-1/a$, $t_2=-1/b$, $t=\infty$ and an irregular at $t=0$. For its solutions around $t=0$ we put:

$$\Psi = e^{m/t}\sum_{n=o}^{\infty} g_n t^{n+\beta} \quad , \quad g_o = 1.$$

Then:

$$\frac{d\Psi}{dt} = e^{m/t}\sum_{n=o}^{\infty}(n+\beta)g_n t^{n+\beta-1} - me^{m/t}\sum_{n=o}^{\infty} g_n t^{n+\beta-2}$$

$$\frac{d^2\Psi}{dt^2} = e^{m/t}\sum_{n=o}^{\infty}(n+\beta)(n+\beta-1)g_n t^{n+\beta-2} - 2me^{m/t}\sum_{n=o}^{\infty}(n+\beta)g_n t^{n+\beta-3} +$$

$$+ m^2e^{m/t}\sum_{n=o}^{\infty} g_n t^{n+\beta-4} + 2me^{m/t}\sum_{n=o}^{\infty} g_n t^{n+\beta-3} \quad .$$

Substituting in (2-21) and after the factor $e^{m/t}t^{\beta}$ is cancelled we obtain:

$$[abt^2+(a+b)t+1][\sum_{n=o}^{\infty}(n+\beta)(n+\beta-1)g_n t^{n+2}-2m\sum_{n=o}^{\infty}(n+\beta)g_n t^{n+1}+m^2\sum_{n=o}^{\infty}g_n t^n+$$
$$+2m\sum_{n=o}^{\infty}g_n t^{n+1}]+[2abt^2+(a+3b)t+2][\sum_{n=o}^{\infty}(n+\beta)g_n t^{n+2}-m\sum_{n=o}^{\infty}g_n t^{n+1}]+$$
$$+[-abv(v+1)t^4-(a+b)v(v+1)t^3+(a^2-v(v+1))t^2+2at+1]\sum_{n=o}^{\infty}g_n t^n = 0 \ .$$

Equating the coefficients of t^n, $n=0,1,2,3,\ldots$ to zero we have:

For t^0 : $g_o+m^2g_o=0$, or $m^2+1=0$, or $m_1=i$, $m_2=-i$.

For t^1 : $-2m\beta g_o+m^2g_1+(a+b)m^2g_o+2mg_o-2mg_o+2ag_o+g_1=0$, or $2m\beta=c$, $\beta=c/2m$, i.e. $\beta_1=-ic/2$, $\beta_2=ic/2$. So, we obtain two normal solutions around $t=0$:

$$\psi_I(t)\sim e^{i/t}t^{-ic/2}(1+g_1t+g_2t^2+\ldots\ldots) \qquad (2\text{-}22)$$

$$\psi_{II}(t)\sim e^{-i/t}t^{ic/2}(1+g_1't+g_2't^2+\ldots\ldots) \ . \qquad (2\text{-}23)$$

The series in parentheses are asymptotic as $t\to 0$ (9 pp. 168-174 444-445, 10 pp. 69-72). The theory also assures that ψ_I, ψ_{II} are twice differentiable and that the asymptotic expansions for ψ_I', ψ_{II}' (even for ψ_I'', ψ_{II}'') can be obtained by formal differentiation of (2-22) and (2-23), respectively (10 pp. 58-77). Now considering $\psi_I(t)$ and $\psi_{II}(t)$ in connection with (2-20) we observe that if $t\to 0$ along the real axis we are going to have:

$$I(\psi_I,t)\Big|_{t=0}=I(\psi_{II},t)\Big|_{t=0}=0 \quad \text{as long as } \mathrm{Re}\gamma > -1+|\mathrm{Imag}c|/2 \ .$$

Since any solution of (2-21) is a linear combination of ψ_I and ψ_{II} we have:

$$\text{If} \quad \mathrm{Re}\gamma > -1+|\mathrm{Imag}c|/2 \ , \quad I[\psi(t),t]\Big|_{t=0}=0 \qquad (2\text{-}24)$$

for any solution $\psi(t)$ of (2-21) and as long as t goes to 0 along

the real t-axis. This statement will serve to fix the path ℓ in (2-13) so that to provide a solution of the difference equation (2-5).

The indicial equation around $t_1=-1/a$ is:

$$\beta(\beta-1)-\left[\frac{t\phi_1(t)}{abt^4(t+1/b)}\right]_{t=-1/a}\beta = 0 \quad \text{with roots } \beta=0 \text{ and } \beta_1= 2.$$

Around $t_2=-1/b$: $\beta(\beta-1)-\left[\frac{t\phi_1(t)}{abt^4(t+1/a)}\right]_{t=-1/b}\beta = 0$, roots: $\beta=0$ and $\beta_2= 0$.

Since $t_1=-1/a$ and $t_2=-1/b$ are regular singular points of (2-21) there exist one solution $\psi_1(t)$ of this equation around $t_1=-1/a$ and another $\psi_2(t)$ around $t_2=-1/b$, which, in the neighborhood of the corresponding point and up to the nearest singularity, can be expressed by the following convergent series:

$$\psi_1(t) = (t+1/a)^2\,[1+p_1(t+1/a)+p_2(t+1/a)^2+\ldots..] \tag{2-25}$$

$$\psi_2(t) = 1+q_1(t+1/b)+q_2(t+1/b)^2+\ldots.. \quad . \tag{2-26}$$

We observe that $\psi_1(-1/a) = \psi_1'(-1/a) = 0$ so that from (2-20): $I[\psi_1(t),t]\big|_{t=-1/a}= 0$. Also from (2-20) and (2-26) we see that: $I[\psi_2(t),t]\big|_{t=-1/b}= 0$. Combining these results with (2-24) we conclude that there exist two corresponding solutions of (2-5) in the form:

$$v_1(y) = \frac{1}{2\pi i}\int_{\ell_1} t^{y-1}\psi_1(t)\,dt \tag{2-27}$$

$$v_2(y) = \frac{1}{2\pi i}\int_{\ell_2} t^{y-1}\psi_2(t)\,dt \quad , \tag{2-28}$$

where the paths ℓ_1 and ℓ_2 are shown in figure (2-1). The dotted lines represent the branch lines of $\psi_1(t)$ and $\psi_2(t)$. The paths ℓ_1 and ℓ_2 can be deformed as long as they do not cross branch lines and end at t=0 along the real t-axis.

The so defined solutions $v_1(y)$ and $v_2(y)$ are analytic for (7 Chapt. VIII, 17 pp. 478-501, 18 Chapt. III):

$$\mathrm{Re}\,y > -1+|\mathrm{Imag}\,c|/2 \quad . \tag{2-29}$$

It will be shown later that they are also independent.

In order to obtain the third and fourth independent solutions of (2-5) we put (7 Chapt. VIII):

$$v(y) = w(y)/\Gamma(y) \tag{2-30}$$

and substituting in (2-5) we obtain:

$$\sum_{m=o}^{4} p_m(y)w(y+m)/\Gamma(y+m) = 0 \ , \quad \text{or}$$

$$P_4(y)w(y+4)+P_3(y)w(y+3)+P_2(y)w(y+2)+P_1(y)w(y+1)+ \\ +P_o(y)w(y) = 0 \ , \tag{2-31}$$

where with the use of (2-6) to (2-10):

$$P_4(y)=p_4(y)=ab[(y+4)(y+5)-2(y+4)-v(v+1)] \tag{2-32}$$

$$P_3(y)=p_3(y)(y+3)=(a+b)(y+3)(y+4)(y+5)-(3a+5b)(y+3)(y+4)+ \\ +[a+3b-(a+b)v(v+1)](y+3) \tag{2-33}$$

$$P_2(y)=p_2(y)(y+2)(y+3)=(y+2)(y+3)(y+4)(y+5)-6(y+2)(y+3)(y+4)+ \\ +[a^2+6-v(v+1)](y+2)(y+3) \tag{2-34}$$

$$P_1(y)=p_1(y)(y+1)(y+2)(y+3)=2a(y+1)(y+2)(y+3) \tag{2-35}$$

$$P_o(y)=p_o(y)y(y+1)(y+2)(y+3)=y(y+1)(y+2)(y+3) \quad . \tag{2-36}$$

Following similar steps as before, we assume a solution in the form: $w(y) = \frac{1}{2\pi i}\int_{\ell} t^{y-1}\psi(t)\,dt$ and form the functions:

$$\phi_4(t) = t^2+1 \tag{2-37}$$

$$\phi_3(t) = (a+b)t^3-6t^2+2at \tag{2-38}$$

$$\phi_2(t) = -2abt^4-(3a+5b)t^3+[a^2+6-v(v+1)]t^2 \tag{2-39}$$

$$\phi_1(t) = -2abt^4+[a+3b-(a+b)v(v+1)]t^3 \tag{2-40}$$

$$\phi_o(t) = -abv(v+1)t^4 \quad . \tag{2-41}$$

Then $\Psi(t)$ must satisfy the following differential equation:

$$(t^2+1)t^4\Psi^{IV}(t)-\phi_3(t)t^3\Psi'''(t)+\phi_2(t)t^2\Psi''(t)-\phi_1(t)\Psi'(t)+$$
$$+\phi_o(t)\Psi(t) = 0 \quad , \tag{2-42}$$

while in this case (17 pp. 478-501, 18 Chapt. III):

$$I(\Psi,t)=\Psi(t)\sum_{m=o}^{3}\frac{d^m}{dt^m}[t^{y+m}\phi_{m+1}(t)]-\Psi'(t)\sum_{m=o}^{2}\frac{d^m}{dt^m}[t^{y+m+1}\phi_{m+2}(t)]+$$
$$+\Psi''(t)\sum_{m=o}^{1}\frac{d^m}{dt^m}[t^{y+m+2}\phi_{m+3}(t)]-\Psi'''(t)t^{y+3}\phi_4(t) \quad . \tag{2-43}$$

Now, equation (2-42) has three regular singular points at $t=0$, $t_3=1$, $t_4=-1$ and an irregular singularity at $t=\infty$. The indicial equation around $t=0$, according to (2-37)-(2-42), is: $\rho(\rho-1)(\rho-2)(\rho-3) = 0$ with roots 0, 1, 2, 3. Four independent regular solutions of (2-42) correspond to these roots and any solution of it can be expressed as a linear combination of these four. These solutions are of the general form:

$\Psi_s(t)=t^{\rho_s}[f_o(t)+f_1(t)\ln t+\ldots\ldots+f_m(t)(\ln t)^m]$, where $s=1,2,3,4$ and $m\leqq 3$. $f_o(t)$, $f_1(t)$, ..., $f_m(t)$ are analytic at $t=0$. Then $t^y\Psi_s(t)$ vanishes at $t=0$, provided that $Re(y+\rho_s)>0$, or $Re y>0,-1,-2,-3$. So, if $Re y>0$, $t^y\Psi_s(t)$ vanishes at $t=0$ for all $\Psi_s(t)$. This fact,

combined with (2-43) means that:

$$\text{If}\quad \mathrm{Re}\,y > 0 :\quad I(\psi,t)\Big|_{t=0} = 0 \tag{2-44}$$

for any solution $\psi(t)$ of (2-43). The result is that for paths ℓ starting and ending at t=0, we will have $[I(\psi,t)]_\ell = 0$ for any solution $\psi(t)$ of (2-43), provided that $\mathrm{Re}\,y > 0$.

The indicial equation around $t_3 = i$ is:

$$\beta(\beta-1)(\beta-2)(\beta-3) - \Big[[t^3\phi_3(t)]/[t^4(t+i)]\Big]_{t=i}\beta(\beta-1)(\beta-2) = 0$$

with roots 0, 1, 2 and $\beta_3 = -ic/2$. Around $t_4 = -i$:

$$\beta(\beta-1)(\beta-2)(\beta-3) - \Big[[t^3\phi_3(t)]/[t^4(t-i)]\Big]_{t=-i}\beta(\beta-1)(\beta-2) = 0$$

with roots 0, 1, 2 and $\beta_4 = ic/2$.

As before, there exist two solutions $\psi_3(t)$ and $\psi_4(t)$ of equation (2-42), the first around t=i, the second around t=-i, which can be expressed by the following convergent series:

$$\psi_3(t) = (t-i)^{-ic/2}[1 + h_1(t-i) + h_2(t-i)^2 + \ldots]\ ,\quad |t-i| < 1 \tag{2-45}$$

$$\psi_4(t) = (t+i)^{ic/2}[1 + f_1(t+i) + f_2(t+i)^2 + \ldots.]\ ,\quad |t+i| < 1\ . \tag{2-46}$$

Then, combining with (2-30), we find two other independent solutions of (2-5) in the form:

$$v_3(y) = \frac{1}{2\pi i\Gamma(y)}\int_{\ell_3} t^{y-1}\psi_3(t)\,dt \tag{2-47}$$

$$v_4(y) = \frac{1}{2\pi i\Gamma(y)}\int_{\ell_4} t^{y-1}\psi_4(t)\,dt\ , \tag{2-48}$$

where ℓ_3 and ℓ_4 are shown in figure (2-2) (7 Chapt. VIII). They start and end at t=0, as condition (2-44) requires, enclosing i and -i counterclockwise. The t-plane is properly cut by the branch lines of $\psi_3(t)$ and $\psi_4(t)$, correspondingly, shown by dotted lines in figure (2-2). The so defined solutions $v_3(y)$ and $v_4(y)$

of (2-5) are analytic for

$$\mathrm{Re}\,y > 0 \quad . \tag{2-49}$$

We have found four independent solutions of the fourth-order linear difference equation (2-5): $v_1(y)$ and $v_2(y)$ defined and analytic for $\mathrm{Re}\,y > -1+|\mathrm{Imag}\,c|/2$ and given by (2-27) and (2-28); $v_3(y)$ and $v_4(y)$ defined and analytic for $\mathrm{Re}\,y > 0$ and given by (2-47), (2-48). For $\mathrm{Re}\,y < -1+|\mathrm{Imag}\,c|/2$ and $\mathrm{Re}\,y < 0$ we define the analytic continuation of $v_1(y)$, $v_2(y)$ and $v_3(y)$, $v_4(y)$, respectively, with the use of the difference equation (2-5) itself. That is:

$$v_s(y) = -[p_4(y)v_s(y+4)+p_3(y)v_s(y+3)+p_2(y)v_s(y+2)+ \\ +2av_s(y+1)] \quad ; \quad s=1,2,3,4 \, . \tag{2-50}$$

We see that the so defined four solutions of (2-5) are analytic in the whole y-plane.

We are now going to obtain expansions of these functions in terms of inverse factorial series and investigate their behaviour as $y \to \infty$ in the right half y-plane, heretofore indicated as: $y \xrightarrow[\text{rhp}]{} \infty$. We first observe that $v_1(y)$, $v_2(y)$, $w_3(y)=v_3(y)\Gamma(y)$, and $w_4(y)=v_4(y)\Gamma(y)$ can be expressed in the general form:

$$u_s(y) = \frac{1}{2\pi i}\int_{\ell_s} t^{y-1}(t-t_s)^{\beta_s} f_s(t)\, dt \quad , \tag{2-51}$$

where $f_s(t)$ is analytic at $t=t_s$ but not at $t=0$; the so defined functions of y are analytic for $\mathrm{Re}\,y > y_s$, (where $y_s=-1+|\mathrm{Imag}\,c|/2$ for s=1,2 and 0 for s=3,4). Looking at equations (2-21) and (2-42) we see that t=0 is a regular singular point for $f_3(t)$ and $f_4(t)$ appearing in the definition (2-51) for $w_3(y)$ and $w_4(y)$, but an irregular singularity of $f_1(t)$ and $f_2(t)$ associated with $v_1(y)$ and $v_2(y)$. (Equation (2-42) has also an irregular singular point at $t=\infty$, but all the paths ℓ_s in (2-51) begin and end at t=0

without approaching the vicinity of $t=\infty$; actually, this is why (2-51) defines $u_s(y)$ only in the right half plane $\mathrm{Re}\,y > y_s$). The appearance of irregular singularities in (2-21) and (2-42) is due originally to the fact that equation (I 1-50) for $R_1(z)$ and $R_2(z)$, and consequently the functions themselves, have an irregular singularity at $z=\infty$ (7 Chapt. VIII); this, in turn, leads to a non-normal form for the associated difference equation (2-5) (7 Chapt. VIII, 17 pp. 478-501, 18 Chapt. III).

When $t=0$ is a regular singular point of $f_s(t)$, it is a familiar fact of the theory of difference equations that, if a sufficiently large positive number ω is selected, then $u_s(y)$, as defined by (2-51) for $\mathrm{Re}\,y > y_s$, except for a constant factor depending only on ω, t_s, β_s, can be expressed in the following form:

$$u_s(y) = t_s^y \frac{\Gamma(y/\omega)}{\Gamma(y/\omega+\beta_s+1)} \Omega^{(s)}_{(y)} \quad , \tag{2-52}$$

where:

$$\Omega^{(s)}_{(y)} = 1+\sum_{n=1}^{\infty} \frac{g_n^{(s)}}{(y+\omega\beta_s+\omega)(y+\omega\beta_s+2\omega)\ldots\ldots(y+\omega\beta_s+n\omega)} \tag{2-53}$$

is an inverse factorial series convergent for $\mathrm{Re}\,y > y_s$ (7 Chapt. VIII, 17 pp. 485-487, 18 pp. 61-64). We can apply these expansions immediately to $w_3(y)$ and $w_4(y)$, since $t=0$ is simply a regular singularity for $f_3(t)$ and $f_4(t)$. Looking at figure (2-2) we see that ℓ_3 and ℓ_4 can be deformed into the circles $|t-i|=1$ and $|t+i|=1$, respectively; referring to references 17 (pp. 485-487) and 18 (pp. 61-64) we conclude that in these cases we can take $\omega = 1$. $v_3(y)$ and $v_4(y)$ are defined by (2-45) to (2-48). Comparing with (2-51) to (2-53) and with $\omega=1$ we obtain, except for a constant factor, the following expansions:

$$v_3(y) = (i)^y\, \Omega^{(3)}_{(y)}/\Gamma(y+1-ic/2) \;;\quad \mathrm{Re}\,y > 0 \;;\; i=e^{i\pi/2} \tag{2-54}$$

$$v_4(y) = (-i)^y \Omega^{(4)}_{(y)}/\Gamma(y+1+ic/2) \quad ; \quad \text{Re}y>0 \quad ; \quad -i=e^{-i\pi/2} \qquad (2\text{-}55)$$

$$\Omega^{(3)}_{(y)} = 1+ \frac{g_1^{(3)}}{y+1-ic/2} + \frac{g_2^{(3)}}{(y+1-ic/2)(y+2-ic/2)} +\ldots \; , \; \text{Re}y>0 \qquad (2\text{-}56)$$

$$\Omega^{(4)}_{(y)} = 1+ \frac{g_1^{(4)}}{y+1+ic/2} + \frac{g_2^{(4)}}{(y+1+ic/2)(y+2+ic/2)} +\ldots \; , \; \text{Re}y>0 \; . \qquad (2\text{-}57)$$

The inverse factorial series $\Omega^{(3)}_{(y)}$, $\Omega^{(4)}_{(y)}$ converge uniformly for Re$y>0$. Their coefficients $g_n^{(3)}$, $g_n^{(4)}$ can be obtained by the method explained in references 17 (Chapt. XV), 18 (Chapt. III), or by direct substitution into the difference equation (2-5) and application of the method of undetermined coefficients. We will find them explicitly later using the former method. For the general theory of inverse factorial series and their properties we refer to references 17 (Chapt. X), 19 (Chapt. VI pp. 170-177).

The constant factor for $v_3(y)$, $v_4(y)$ was selected so as to give them the form indicated in (2-54) to (2-57), which will prove of convenience later. Hereafter we adopt these definitions for $v_3(y)$ and $v_4(y)$, instead of the integral forms (2-47),(2-48). They are simply constant multiples of the latter. As before, $v_3(y)$ and $v_4(y)$ are analytic for Re$y>0$, the same being true for $\Omega^{(3)}_{(y)}$ and $\Omega^{(4)}_{(y)}$. Their analytic continuation for Re$y<0$ is again provided by (2-50) and makes the so defined functions $v_3(y)$ and $v_4(y)$ analytic throughout the y-plane.

When $t=0$ is an irregular singular point of $f_s(t)$ in (2-51), convergent expansions of the form (2-52) and (2-53) can not, in general, be obtained. However, with $\omega=1$ (2-52) and (2-53) provide an asymptotic expansion for $u_s(y)$, as defined in (2-51), as $y\to\infty$ in the sector $-\pi/2+\epsilon < \arg y < \pi/2+\epsilon$ ($\epsilon>0$ and arbitrarily small) (7 pp. 309-318, 8 pp. 70-74, 17 pp. 457-459, 20). This statement

is based upon a theorem contained in reference 8 (pp. 70-74). It is due originally to Nörlund and its proof can be found in Nörlund's original paper in "Acta Mathematica", reference 20. We give the theorem in full, as contained in reference 8 by Ford, because it serves to clarify many points regarding the solutions of difference equations with polynomial coefficients, which we have been making use of.

<u>Nörlund's Theorem:</u> <u>"Given the linear difference equation</u>

$$\sum_{i=0}^{k} P_i(x)u(x-i) = 0 \quad , \tag{I}$$

where the coefficients are factorial series of the form:

$$P_i(x)=c_0^{(i)}+\frac{c_1^{(i)}}{x+1}+\frac{c_2^{(i)}}{(x+1)(x+2)}+\frac{c_3^{(i)}}{(x+1)(x+2)(x+3)}+\ldots\ , \tag{II}$$

$$i = 0,1,2,\ldots\ldots,k \quad ,$$

all of which converge throughout the right half of the x-plane. Suppose first that the roots a_1, a_2, ..., a_k of the characteristic equation

$$c_0^{(0)}z^k+c_0^{(1)}z^{k-1}+\ldots\ldots+c_0^{(k)}=0 \quad ; \quad c_0^{(0)}\neq 0\ ,\ c_0^{(k)}\neq 0 \tag{III}$$

are distinct. Then there exist k solutions u_1, u_2,....., u_k of (I) such that throughout the sector $-\pi/2+\epsilon < \arg x < \pi/2-\epsilon$ (ϵ arbitrarily small and >0) we have:

$$u_j \sim a_j^x \frac{\Gamma(x+1)}{\Gamma(x-\rho_j+1)}\phi_j(x) \quad , \tag{IV}$$

where ρ_j is a constant and $\phi_j(x)$ a factorial series of the form indicated in (II). In case (III) has multiple roots and a_j is an n-fold root, two cases are distinguished:

(1) a_j is at the same time an (n-p)-fold root of the equations

$$\sum_{s=0}^{k} c_p^{(s)}z^{k-s} = 0 \ ; \quad p = 1,2,\ldots, n-1 \ .$$

(2) These conditions are not fullfiled.

In (2) no asymptotic development exists of the form (IV). In (1) there exist n linearly independent solutions $u_s(x)$ $(s=1,2,\ldots,n)$ such that when $-\pi/2+\epsilon < \arg x < \pi/2-\epsilon$ we have:

$$u_s \sim a_j \Phi_s(x) \quad ; \quad s=1,2,\ldots,n \quad ,$$

where:

$$\Phi_s(x) = \phi_0(x)\frac{\Gamma(x+1)}{\Gamma(x-\rho_s+1)} + \phi_1(x)\frac{\partial}{\partial\rho_s}\frac{\Gamma(x+1)}{\Gamma(x-\rho_s+1)} + \cdots + \phi_n(x)\frac{\partial^n}{\partial\rho_s^n}\frac{\Gamma(x+1)}{\Gamma(x-\rho_s+1)}$$

the expressions $\phi_0, \phi_1, \ldots, \phi_n$ being developments of the form (II).

If some of the roots of (III) are zero or infinite, it is necessary, in order to obtain a system of fundamental solutions, to use a series of substitutions of the form:

$$u(x) = [\Gamma(x)]^{\mu_r} w_{(x)}^{(\mu_r)} = \Gamma_{(x)}^{\mu_r} \quad u_{(x)}^{(\mu_r)}$$

and determine μ_r so that the difference equation in $w_{(x)}^{(\mu_r)}$ shall have a characteristic equation containing at least one root which is finite and different from 0. It is always possible to determine in but one way, a series of numbers $\mu_1, \mu_2, \ldots, \mu_m$ such that the total number of roots which are finite and different from zero in the corresponding characteristic equations thus obtained, is exactly the order k of (I). If, whenever a multiple root occurs in one of these characteristic equations, the corresponding conditions under (1) are satisfied, then there exist a system of fundamental solutions of (I), each of which is asymptotically represented within the sector $-\pi/2+\epsilon < \arg x < \pi/2-\epsilon$ by a series of the form: $\Gamma_{(x)}^{\mu_r} \; a_j^x \; \Phi_s(x)$. If no multiple roots occur in one of these characteristic equations, the solutions have the simpler form:

$$\Gamma_{(x)}^{\mu_r} a_j^x \frac{\Gamma(x+1)}{\Gamma(x-\rho_j+1)} \phi_j(x) .$$

Exceptions occur when some of the numbers μ_r are not integers. Then the coefficients in the above-mentioned difference equations for $w_{(x)}^{(\mu_r)}$ are no longer developable in factorial series of the form (II). Suppose μ_r is equal to a rational fraction p/q. We put x=pz, w(x)=v(z) and derive from (I) a difference equation for v(z), thus demonstrating the existence of solutions expressible asymptotically in the forms:

$$\Gamma_{(x/p)}^{\mu_r} a_j^{x/p} \phi_s(x/p) ."$$

Here we are dealing with the difference equation (2-5). Dividing by $p_4(y)$ and making appropriate change of variables, like $y+\tau=x$, $v(y)=u(x)$, we can reduce it to the forms (I), (II). We can then easily verify the statement made on page 2-16.

Returning now to (2-51) we observe that $f_1(t)$ and $f_2(t)$, appearing in the definitions of $v_1(y)$, $v_2(y)$ for $\mathrm{Re}\,y > -1+|\mathrm{Imag}\,c|/2$, have an irregular singularity at t=0. Then, according to the preceding discussion, (2-52) and (2-53) with $\omega=1$ provide at least asymptotic expansions for $v_1(y)$ and $v_2(y)$ when $y \underset{\mathrm{rhp}}{\longrightarrow} \infty$. Referring to figure (2-1) and to the definitions (2-25) to (2-28), we obtain, except for a constant factor, the following expansions:

$$v_1(y)=(-1/a)^y \frac{\Gamma(y)}{\Gamma(y+3)} \Omega_{(y)}^{(1)} = (-1/a)^y \frac{\Omega_{(y)}^{(1)}}{y(y+1)(y+2)} \tag{2-58}$$

$$v_2(y)=(-1/b)^y \frac{\Gamma(y)}{\Gamma(y+1)} \Omega_{(y)}^{(2)} = (-1/b)^y \,\Omega_{(y)}^{(2)}/y \tag{2-59}$$

$$\Omega_{(y)}^{(1)} \underset{y \underset{\mathrm{rhp}}{\longrightarrow} \infty}{\sim} 1+ \frac{g_1^{(1)}}{y+3} + \frac{g_2^{(1)}}{(y+3)(y+4)} + \cdots\cdots \tag{2-60}$$

$$\Omega^{(2)}_{(y)} \underset{y \xrightarrow[rhp]{} \infty}{\sim} 1+\frac{g_1^{(2)}}{y+1}+\frac{g_2^{(2)}}{(y+1)(y+2)}+\cdots\cdots . \tag{2-61}$$

All these expressions are valid in the sector $-\pi/2+\epsilon < \arg y < \pi/2-\epsilon$. The inverse factorial series $\Omega^{(1)}_{(y)}$, $\Omega^{(2)}_{(y)}$ are at least asymptotic as $y \xrightarrow[rhp]{} \infty$ in the indicated sector. Their coefficients $g_n^{(s)}$, $s=1,2$, can be obtained by both methods mentioned previously for $g_n^{(3)}$, $g_n^{(4)}$, the only difference being that the process is now at least formal. As we shall see, it will not be necessary to find them explicitly. Hereafter we adopt the above definitions for $v_1(y)$, $v_2(y)$. They are simply constant multiples of the integral forms (2-27),(2-28) of the analytic functions $v_1(y)$, $v_2(y)$.

The linear independence of the so obtained solutions $v_1(y)$, $v_2(y)$, $v_3(y)$, $v_4(y)$ can be shown by referring to Nörlund's theorem, or by making use of a general theorem given in reference 17 (p. 360), based upon the behaviour of the solutions $v_s(y)$ ($s=1,2,3,4$) as $y \xrightarrow[rhp]{} \infty$, which we now know.

We have obtained for $v_3(y)$, $v_4(y)$ two convergent and for $v_1(y)$, $v_2(y)$ two at least asymptotic expansions in terms of inverse factorial series valid in a half plane limited to the left. It is a fact of the theory of linear difference equations that when $|y|$ is large (without any limitation to the left), the corresponding expansions $\Omega^{(s)}_{(y)}$, though not necessarily convergent any more, are in any case developable asymptotically in series of the same form (7 pp. 309-318, 17 pp. 457-459). Thus, we have:

$$v_1(y) = (-1/a)^y \frac{1+\epsilon_1(y)}{y(y+1)(y+2)} \tag{2-62}$$

$$v_2(y) = (-1/b)^y \frac{1+\epsilon_2(y)}{y} \tag{2-63}$$

$$v_3(y) = (i)^y \Omega^{(3)}_{(y)}/\Gamma(y+1-ic/2) \quad ; \quad i=e^{i\pi/2} \tag{2-64}$$

$$v_4(y) = (-i)^y \Omega^{(4)}_{(y)}/\Gamma(y+1+ic/2) \quad ; \quad -i=e^{-i\pi/2} \quad , \tag{2-65}$$

where:

$$\lim_{|y|\to\infty} \epsilon_s(y) = 0 \quad , \quad s=1,2 \tag{2-66}$$

$$\Omega^{(s)}_{(y)} \underset{|y|\to\infty}{\sim} 1+ \frac{g_1^{(s)}}{y+1\mp ic/2} + \frac{g_2^{(s)}}{(y+1\mp ic/2)(y+2\mp ic/2)} +\ldots\ldots, \quad s=3,4 \tag{2-67}$$

where the upper sign should be used for s=3, the lower for s=4.

ASYMPTOTIC EXPANSIONS OF $R_1(z)$, $R_2(z)$ FOR LARGE $|z|$

We are now in a position to apply these results and Theorems I and VI to obtain the asymptotic expansions of $R_1(z)$, $R_2(z)$ (for the latter when $2v+1$ is not equal to an integer), solutions of the differential equation (I 1-50). As was stated in the preceding section, we are looking for the particular solution $\bar{v}(y)$ of (2-5), which satisfies the conditions

$$\bar{v}(n+\sigma) = a_n \quad \text{for all integers } n \tag{2-68}$$

and where for $R_1(z)$ we use $\sigma=\sigma_1=v+1$, while for $R_2(z)$ (for non-integral values of $2v+1$) $\sigma=\sigma_2=-v$. We have obtained four independent solutions of the fourth order difference equation (2-5). Then:

$$\bar{v}(y) = Av_1(y)+Bv_2(y)+Cv_3(y)+Dv_4(y) \quad , \tag{2-69}$$

where A, B, C, D are constants. It is now obvious that with $\sigma=v+1$, or $\sigma=-v$ (if $2v+1$ is not an integer) and the following four initial conditions:

$$\bar{\bar{v}}(0+\sigma) = \bar{\bar{v}}(\sigma) = 1 \tag{2-70}$$

$$\bar{\bar{v}}(-1+\sigma) = 0 \tag{2-71}$$

$$\bar{\bar{v}}(-2+\sigma) = 0 \tag{2-72}$$

$$\bar{\bar{v}}(-3+\sigma) = 0 \quad , \tag{2-73}$$

$\bar{\bar{v}}(y+\sigma)$ will satisfy the conditions (2-68) for all integral values of n. The proof is as follows: For $n=0,1,2,3,\ldots.$ and by virtue of the above relations, $\bar{\bar{v}}(n+\sigma)$ will satisfy the same recurrence formula (1-8), which the coefficients a_n satisfy, since for $y=n+\sigma$ $(n=0,1,2,\ldots)$ the difference equation (2-5) reduces to the recurrence formula (1-8); in fact, (2-5) was derived from (1-8) in the preceding section. Thus we have: $\bar{\bar{v}}(n+\sigma)=a_n$, $n=0,1,2,3,\ldots$, verifying (2-68) for such n's. On the other hand, for $y=-4+\sigma$ (2-5) yields:

$$p_4(-4+\sigma)\bar{\bar{v}}(\sigma)+p_3(-3+\sigma)\bar{\bar{v}}(-1+\sigma)+p_2(-2+\sigma)\bar{\bar{v}}(-2+\sigma)+2a\bar{\bar{v}}(-3+\sigma)+\bar{\bar{v}}(-4+\sigma)=0.$$

Using the last three conditions (2-71)-(2-73) we get:

$$p_4(-4+\sigma)+\bar{\bar{v}}(-4+\sigma) = 0.$$

According to (2-6): $p_4(-4+\sigma)=ab(\sigma-v-1)(\sigma+v)$. Since $\sigma=v+1$ or $\sigma=-v$, we see that in either case $p_4(-4+\sigma)=0$ and, consequently, $\bar{\bar{v}}(-4+\sigma)=$ $=0$ $(\sigma=\sigma_1$ or $\sigma=\sigma_2)$. Using (2-5) and $\bar{\bar{v}}(-n+\sigma)=0$ for $n=1,2,3,4$ we see also that $\bar{\bar{v}}(-n+\sigma)=0$ for $n=5,6,7,\ldots$. Therefore, $\bar{\bar{v}}(n+\sigma)=0=a_n$ for $n=-1,-2,-3,-4,-5,\ldots..$ and (2-68) is verified for all integral values of n, as required. The initial conditions (2-70)-(2-73) serve at the same time to determine the coefficinrs A, B, C, D in (2-69).

As a consequence of all these results we can write:

$$R_{\frac{1}{2}}(z) = z^{\sigma}\sum_{n=0}^{\infty} a_n z^n = z^{\sigma}\sum_{n=0}^{\infty} \bar{\bar{v}}(n+\sigma)z^n,$$ where $\sigma=v+1$ is used for $R_1(z)$

and, when 2v+1 is not an integer, $\sigma=-v$ for $R_2(z)$. Using (2-69) for $\bar{v}(n+\sigma)$ we obtain:

$$R_{\substack{1\\2}}(z)= A(-z/a)^{\sigma}\sum_{n=o}^{\infty} G_1(n)(-z/a)^n+B(-z/b)^{\sigma}\sum_{n=o}^{\infty} G_2(n)(-z/b)^n+$$

$$+C(iz)^{\sigma}\sum_{n=o}^{\infty} G_3(n)(iz)^n+D(-iz)^{\sigma}\sum_{n=o}^{\infty} G_4(n)(-iz)^n \quad , \tag{2-74}$$

where $G_s(n) = v_s(n+\sigma)/t_s^{n+\sigma}$, $s=1,2,3,4$; $t_1=-1/a$, $t_2=-1/b$, $t_3=i$, $t_4=-i$; i.e. $G_s(n)$ when considered as functions of $w=x+iy$ are given by:

$$G_1(w)=v_1(w+\sigma)/(-1/a)^{w+\sigma}= \frac{1+\epsilon_1(w+\sigma)}{(w+\sigma)(w+\sigma+1)(w+\sigma+2)} \tag{2-75}$$

$$G_2(w)=v_2(w+\sigma)/(-1/b)^{w+\sigma}= \frac{1+\epsilon_2(w+\sigma)}{w+\sigma} \tag{2-76}$$

$$\lim_{|w|\to\infty} \epsilon_1(w+\sigma) = \lim_{|w|\to\infty} \epsilon_2(w+\sigma) = 0 \quad ,$$

$$G_3(w)=v_3(w+\sigma)/(i)^{w+\sigma}= \Omega^{(3)}_{(w+\sigma)}/\Gamma(w+\sigma+1-ic/2) \tag{2-77}$$

$$G_4(w)=v_4(w+\sigma)/(-i)^{w+\sigma}= \Omega^{(4)}_{(w+\sigma)}/\Gamma(w+\sigma+1+ic/2) \tag{2-78}$$

$$\Omega^{(3)}_{(w+\sigma)} \underset{|w|\to\infty}{\sim} 1+ \frac{g_1^{(3)}}{w+\sigma+1-ic/2} + \frac{g_2^{(3)}}{(w+\sigma+1-ic/2)(w+\sigma+2-ic/2)} +\dots \tag{2-79}$$

$$\Omega^{(4)}_{(w+\sigma)} \underset{|w|\to\infty}{\sim} 1+ \frac{g_1^{(4)}}{w+\sigma+1+ic/2} + \frac{g_2^{(4)}}{(w+\sigma+1+ic/2)(w+\sigma+2+ic/2)} +\dots, \tag{2-80}$$

the last expressions in (2-75)-(2-78), as well as (2-79) and (2-80) being obtained with the use of (2-62)-(2-67). With $\mathrm{Re}(w+\sigma) > 0$, or $\mathrm{Re}w > -\sigma$, (2-79) and (2-80) are convergent series. The functions $G_s(w)$ are analytic for all w just as the functions $v_s(w+\sigma)$ are. Furthermore, according to (2-75)-(2-80), $G_1(w)$ and $G_2(w)$ satisfy all the conditions of Theorem I subject to the

remarks i), ii), while $G_3(w)$ and $G_4(w)$ satisfy all the conditions of Theorem VI with $p_3=\sigma+1-ic/2$ and $p_4=\sigma+1+ic/2$. According to the ratio test and with the help of (2-75)-(2-80) we see that the first series in (2-74) is convergent for $|z|<|a|$, the second for $|z|<|b|$, while the last two are entire functions. The first two can be expanded asymptotically for large $|z|$ by applying Theorem I, the last two by applying Theorem VI. With

$$z = |z|e^{i\phi} \qquad -\pi<\phi\leqq\pi \tag{2-81}$$

we have using Theorem I, i), ii):

$$A(-z/a)^\sigma\sum_{n=o}^{\infty}G_1(n)(-z/a)^n\sim -A(-z/a)^\sigma\sum_{n=1}^{\infty}G_1(-n)/(-z/a)^n=$$

$$=-Az^\sigma\sum_{n=1}^{\infty}G_1(-n)(-1/a)^{-n+\sigma}/z^n\ ,\ \text{for } -\pi<\phi\leqq\pi,\ \phi\neq\arg(-a),$$

or using (2-75):

$$A(-z/a)^\sigma\sum_{n=o}^{\infty}G_1(n)(-z/a)^n\sim -z^\sigma\sum_{n=1}^{\infty}\frac{Av_1(-n+\sigma)}{z^n}\ ,\ -\pi<\phi\leqq\pi\ ,\ \phi\neq\arg(-a)\ . \tag{2-82}$$

Similarly with the use of (2-76):

$$B(-z/b)^\sigma\sum_{n=o}^{\infty}G_2(n)(-z/b)^n\sim -z^\sigma\sum_{n=1}^{\infty}\frac{Bv_2(-n+\sigma)}{z^n}\ ,\ -\pi<\phi\leqq\pi\ ,\ \phi\neq\arg(-b)\ . \tag{2-83}$$

For the third term in (2-74) we apply Theorem VI with the accompanying it remark and with $p=\sigma+1-ic/2$. Also here $i=e^{i\pi/2}$ and so, with (2-81), $-\pi/2<\arg(iz)\leqq 3\pi/2$. Thus:

$$C(iz)^\sigma\sum_{n=o}^{\infty}G_3(n)(iz)^n\sim -(iz)^\sigma\sum_{n=1}^{\infty}\frac{CG_3(-n)}{(iz)^n}\ ;\ \pi/2<\arg(iz)<3\pi/2 \text{ or } 0<\phi<\pi$$

$$\sim -(iz)^\sigma\sum_{n=1}^{\infty}\frac{CG_3(-n)}{(iz)^n}+(iz)^\sigma Ce^{iz}(iz)^{-\sigma+ic/2}\Big[1+\sum_{n=1}^{\infty}\frac{g_n^{(3)}}{(iz)^n}\Big]\ ;$$

$$-\pi/2 < \arg(iz) \leqq \pi/2 \quad \text{or} \quad -\pi < \phi \leqq 0 \ .$$

According to the theorem in the second expansion we must take:

$(iz)^{-\sigma+ic/2}=e^{(-\sigma+ic/2)(\ln|z|+i\psi)}$, where $-\pi < \psi=\arg(iz) \leqq \pi$.

Since in this case $-\pi < \phi \leqq 0$, we must take $\psi=\phi+\pi/2$, because only ~~in this way we can have $-\pi/2 < \psi \leqq \pi/2$, in agreement with $-\pi < \psi \leqq \pi$.~~ Then we obtain:

$$(iz)^{-\sigma+ic/2}=z^{-\sigma+ic/2}e^{i\pi(-\sigma+ic/2)/2}=z^{-\sigma+ic/2}i^{-\sigma}e^{-\pi c/4} \ . \qquad (2\text{-}84)$$

Using (2-77) and (2-84) we finally obtain:

$$C(iz)^{\sigma}\sum_{n=0}^{\infty} G_3(n)(iz)^n \sim -z^{\sigma}\sum_{n=1}^{\infty}\frac{Cv_3(-n+\sigma)}{z^n} \ , \quad 0 < \phi < \pi$$

$$\sim -z^{\sigma}\sum_{n=1}^{\infty}\frac{Cv_3(-n+\sigma)}{z^n}+Ce^{-\pi c/4}e^{iz}z^{ic/2}\left[1+\sum_{n=1}^{\infty}\frac{g_n^{(3)}}{(iz)^n}\right] \ , \ -\pi < \phi \leqq 0 \qquad (2\text{-}85)$$

For the last term we have: $-i=e^{-i\pi/2}$, $-\pi < \phi \leqq \pi$, $-3\pi/2 < \arg(-iz) \leqq \pi/2$, $p=\sigma+1+ic/2$. Then:

$$D(-iz)^{\sigma}\sum_{n=0}^{\infty} G_4(n)(-iz)^n \sim -(-iz)^{\sigma}\sum_{n=1}^{\infty}\frac{DG_4(-n)}{(-iz)^n} \ , \ -3\pi/2 < \arg(-iz) < -\pi/2 \ \text{or} \ -\pi < \phi < 0$$

$$\sim -(-iz)^{\sigma}\sum_{n=1}^{\infty}\frac{DG_4(-n)}{(-iz)^n}+(-iz)^{\sigma}De^{-iz}(-iz)^{-\sigma-ic/2}\left[1+\sum_{n=1}^{\infty}\frac{g_n^{(4)}}{(-iz)^n}\right] \ ,$$

$$-\pi/2 \leqslant \arg(-iz) \leqq \pi/2 \quad \text{or} \quad 0 \leqq \phi \leqq \pi \ .$$

In this case: $(-iz)^{-\sigma-ic/2}= e^{(-\sigma-ic/2)(\ln|z|+i\psi)}$, where $-\pi < \psi=\arg(-iz) \leqq \pi$. With $0 \leqq \phi \leqq \pi$ we must take $\psi=\phi-\pi/2$, resulting in $-\pi/2 \leqq \psi \leqq \pi/2$, in agreement with $-\pi < \psi \leqq \pi$. Thus:

$$(-iz)^{-\sigma-ic/2}=z^{-\sigma-ic/2}e^{(-\sigma-ic/2)(-i\pi/2)}= z^{-\sigma-ic/2}(-i)^{-\sigma}e^{-\pi c/4}.$$

This result and (2-78) finally yield:

$$D(-iz)^{\sigma}\sum_{n=o}^{\infty} G_4(n)(-iz)^n \sim -z^{\sigma}\sum_{n=1}^{\infty}\frac{Dv_4(-n+\sigma)}{z^n} +$$

$$+De^{-\pi c/4}e^{-iz}z^{-ic/2}[1+\sum_{n=1}^{\infty}\frac{g_n^{(4)}}{(-iz)^n}] \ , \ 0\leqq\phi\leqq\pi$$

$$\sim -z^{\sigma}\sum_{n=1}^{\infty}\frac{Dv_4(-n+\sigma)}{z^n} \ , \quad -\pi<\phi<0 \ . \tag{2-86}$$

Combining (2-82),(2-83),(2-85) and (2-86), substituting in (2-74) and making use of (2-69) we obtain:

$$R_{\substack{1\\2}}(z)\sim -z^{\sigma}\sum_{n=1}^{\infty}\frac{\bar{\bar{v}}(-n+\sigma)}{z^n} +De^{-\pi c/4-iz} \ z^{-ic/2}[1+\sum_{n=1}^{\infty}\frac{g_n^{(4)}}{(-iz)^n}] \ , \ 0<\phi<\pi$$

$$-z^{\sigma}\sum_{n=1}^{\infty}\frac{\bar{\bar{v}}(-n+\sigma)}{z^n} +Ce^{-\pi c/4+iz} \ z^{ic/2}[1+\sum_{n=1}^{\infty}\frac{g_n^{(3)}}{(iz)^n}] \ , \ -\pi<\phi<0$$

$$-z^{\sigma}\sum_{n=1}^{\infty}\frac{\bar{\bar{v}}(-n+\sigma)}{z^n} +Ce^{-\pi c/4+iz} \ z^{ic/2}[1+\sum_{n=1}^{\infty}\frac{g_n^{(3)}}{(iz)^n}]+$$

$$+De^{-\pi c/4-iz}z^{-ic/2}[1+\sum_{n=1}^{\infty}\frac{g_n^{(4)}}{(-iz)^n}] \ , \ \phi=0 \ ; \ \phi\neq\arg(-a,-b) \tag{2-87}$$

The ray $\phi=\pi$ is a branch line for $R_1(z)$, $R_2(z)$ and is excluded from the above expansions. For large $|z|$ we can draw the branch lines, starting from $z=-a$ and $z=-b$, along $\phi=\pi$. Also according to (2-68): $\bar{\bar{v}}(-n+\sigma) = a_{-n} = 0$, $n=1,2,3,\ldots$, so that we finally obtain:

$$R_{\substack{1\\2}}(z)\sim De^{-\pi c/4} \ e^{-iz} \ z^{-ic/2} \ [1+\sum_{n=1}^{\infty}\frac{g_n^{(4)}}{(-iz)^n}] \ , \ 0<\phi<\pi$$

$$\sim Ce^{-\pi c/4} \ e^{iz} \ z^{ic/2} \ [1+\sum_{n=1}^{\infty}\frac{g_n^{(3)}}{(iz)^n}] \ , \ -\pi<\phi<0$$

$$\sim Ce^{-\pi c/4}\, e^{iz}\, z^{ic/2}\left[1+\sum_{n=1}^{\infty}\frac{g_n^{(3)}}{(iz)^n}\right]+$$

$$+De^{-\pi c/4}\, e^{-iz}\, z^{-ic/2}\left[1+\sum_{n=1}^{\infty}\frac{g_n^{(4)}}{(-iz)^n}\right]\ ,\ \phi = 0\ . \qquad (2\text{-}88)$$

For $R_1(z)$ we use $\sigma=v+1$, for $R_2(z)$ (when $2v+1$ is not an integer), $\sigma=-v$.

In order to complete the problem we are going next to determine explicitly the coefficients $g_n^{(3)}$, $g_n^{(4)}$ and identify the above expansions with the previously determined asymptotic solutions $R_3(z)$, $R_4(z)$. Anticipating the forthcoming proof, we can state that, in an entirely independent manner, we have arrived at the required expansion (1-97) for $R_1(z)$, $R_2(z)$, in complete agreement with the results of the preceding chapter and, at the same time, have obtained explicit relations for the precise evaluation of the coefficients of the linear relations.

For $g_n^{(3)}$, $g_n^{(4)}$ we use the method explained in references 17 (Chapt. XV) and 18 (Chapt. III), mentioned on page 2-16. We first determine the coefficients h_n of the solution $\Psi_3(t)$ around $t=i$ of equation (2-42). $\Psi_3(t)$ is given by (2-45). We put $t-i=z$, $t=z+i$ and substitute in (2-37)-(2-42):

$$z(z+2i)\frac{d^4\Psi}{dz^4} -[(a+b)(z+i)^2-6(z+i)+2a]\frac{d^3\Psi}{dz^3} +[ab(z+i)^3-(3a+5b)(z+i)+$$

$$+a^2+6-v(v+1)]\frac{d^2\Psi}{dz^2} -[-2ab(z+i)+a+3b-(a+b)v(v+1)]\frac{d\Psi}{dz} -$$

$$-abv(v+1)\Psi(z) = 0 \quad . \qquad (2\text{-}89)$$

Assume $\Psi_3(z) = \sum_{n=0}^{\infty} h_n z^{n+\beta}$ and substitute in (2-89). After multiplying it by $z^{3-\beta}$ we obtain:

$$\sum_{n=0}^{\infty}(n+\beta)(n+\beta-1)(n+\beta-2)(n+\beta-3)h_n z^n(z+2i) - \sum_{n=0}^{\infty}(n+\beta)(n+\beta-1)(n+\beta-2)h_n\cdot$$

$$\cdot z^n\left\{(a+b)z^2+2[i(a+b)-3]z+c-6i\right\}+\sum_{n=o}^{\infty}(n+\beta)(n+\beta-1)h_n z^n\left\{abz^3-(3a+5b-\right.$$

$$\left.-2iab)z^2+[ac+6-v(v+1)-i(3a+5b)]z\right\}+\sum_{n=o}^{\infty}(n+\beta)h_n z^n\left\{2abz^3+[(a+b)v(v+1)-\right.$$

$$\left.-a-3b+2iab]z^2\right\}-\sum_{n=o}^{\infty}h_n z^n abv(v+1)z^3=\sum_{n=o}^{\infty}z^n h_n f(z,n+\beta) = 0 \ ,$$

where:

$$f(z,n+\beta) = (n+\beta)(n+\beta-1)(n+\beta-2)(n+\beta-3)(z+2i)-(n+\beta)(n+\beta-1)(n+\beta-2)\cdot$$

$$\cdot\left\{(a+b)z^2+2[i(a+b)-3]z+c-6i\right\}+(n+\beta)(n+\beta-1)\left\{abz^3-(3a+5b-2iab)z^2+\right.$$

$$\left.+[ac+6-v(v+1)-i(3a+5b)]z\right\}+(n+\beta)\left\{2abz^3+[(a+b)v(v+1)-a-3b+2iab]z^2\right\}-$$

$$-abv(v+1)z^3= f_o(n+\beta)+f_1(n+\beta)z+f_2(n+\beta)z^2+f_3(n+\beta)z^3 \quad ,$$

where:

$$f_o(x)=x(x-1)(x-2)[2i(x-3)-c+6i] \tag{2-90}$$

$$f_1(x)=x(x-1)\left\{(x-2)(x-3)+2[3-i(a+b)](x-2)+ac+6-v(v+1)-\right.$$
$$\left.-i(3a+5b)\right\} \tag{2-91}$$

$$f_2(x)=x[-(a+b)(x-1)(x-2)-(3a+5b-2iab)(x-1)+(a+b)v(v+1)-$$
$$-a-3b+2iab] \tag{2-92}$$

$$f_3(x)=ab[x(x-1)+2x-v(v+1)]=ab(x-v)(x+v+1) \quad . \tag{2-93}$$

Recurrence relation:

$$\sum_{m=o}^{3}f_m(n+\beta-m)h_{n-m}= 0 \ ; \ h_o= 1 \ ; \ h_{-j}= 0 \ , \ j=1,2,\dots\ . \tag{2-94}$$

Indicial equation: $f_o(\beta)=\beta(\beta-1)(\beta-2)[2i(\beta-3)-c+6i] = 0$, with roots $\beta = 0,1,2$ and $\beta_3= -ic/2$. Thus, as in (2-45), we have:

$$\psi_3(t) = (t-i)^{-ic/2}\ q(t-i) \tag{2-95}$$

$$q(t-i) = 1+\sum_{n=1}^{\infty} h_n(t-i)^n \quad ; \quad |t-i|<1 \quad . \tag{2-96}$$

We can also write (2-94) in the following form:

$$\sum_{m=o}^{3} F_m(n)h_{n-m} = 0 \quad ; \quad h_o = 1 \; ; \quad h_{-j} = 0 \; , \; j = 1,2,\dots \; , \tag{2-97}$$

where:

$$F_o(n)=f_o(n-ic/2)=2in(n-ic/2)(n-1-ic/2)(n-2-ic/2) \tag{2-98}$$

$$F_1(n)=f_1(n-ic/2-1)=(n-1-ic/2)(n-2-ic/2)[(n-1-ic/2)(n-\tfrac{1}{2}(5a+3b))+ +c(a+i)-v(v+1)] \tag{2-99}$$

$$F_2(n)=f_2(n-ic/2-2)=-(n-2-ic/2)[(a+b)(n-ic/2-2)^2+2b(1-ia)\cdot \cdot(n-2-ic/2)-(a+b)v(v+1)] \tag{2-100}$$

$$F_3(n)=f_3(n-ic/2-3)=ab(n-3-v-ic/2)(n-2+v-ic/2) \quad . \tag{2-101}$$

From (2-47),(2-95),(2-96) we obtain:

$$w_3(y) = \frac{1}{2\pi i}\int_{\ell_3} t^{y-1}(t-i)^{-ic/2}q(t-i)\,dt \quad ,$$

where $q(t-i)$ has a regular singularity at $t=0$. Putting $t=iz$ we get:

$$w_3(y)= \frac{i^y}{2\pi i}\int_{L_3} z^{y-1}\, i^{-ic/2}(z-1)^{-ic/2}q[i(z-1)]\,dz \; , \; \mathrm{Re}\,y>0, \tag{2-102}$$

where L_3, the mapping of the path ℓ_3 in figure (2-2), is shown in figure (2-3). The dotted lines are the branch lines of $q[i(z-1)]$. We put:

$$\frac{i^{-ic/2}}{2\pi i}q[i(z-1)]= \frac{e^{\pi c/4}}{2\pi i}[1+\sum_{n=1}^{\infty}h_n i^n(z-1)^n]=\sum_{n=o}^{\infty}d_n(1-z)^n \; , \tag{2-103}$$

where:

$$d_n = h_n\frac{e^{\pi c/4}}{2\pi}(-i)^{n+1} \quad . \tag{2-104}$$

Then, according to references 17 (Chapt. XV) and 18 (Chapt. III), after substitution in (2-102) and term by term integration, we obtain for Rey > 0:

$$w_3(y)=e^{-\pi i(-ic/2)}i^y[1-e^{2\pi i(-ic/2)}]\sum_{n=o}^{\infty} d_n \frac{\Gamma(y)\Gamma(n+1-ic/2)}{\Gamma(y+n+1-ic/2)} =$$

$$=i^y\frac{\Gamma(y)}{\Gamma(y+1-ic/2)}[i\sinh\frac{\pi c}{2}\,\frac{e^{\pi c/4}}{\pi}\Gamma(1-ic/2)]\sum_{n=o}^{\infty}(-i)^n h_n\,\cdot$$

$$\cdot\,\frac{\Gamma(n+1-ic/2)\Gamma(y+1-ic/2)}{\Gamma(y+n+1-ic/2)\Gamma(1-ic/2)}\quad .$$

The expression in brackets is a constant coefficient. Referring to (2-54),(2-56) we notice that as $v_3(y)=w_3(y)/\Gamma(y)$ we chose:

$$v_3(y) = i^y\,\Omega^{(3)}_{(y)}/\Gamma(y+1-ic/2) \quad , \text{ where:}$$

$$\Omega^{(3)}_{(y)}=1+\frac{g_1^{(3)}}{y+1-ic/2}+\frac{g_2^{(3)}}{(y+1-ic/2)(y+2-ic/2)}+\ldots.=\sum_{n=o}^{\infty}(-i)^n h_n\,\cdot$$

$$\cdot\,\frac{\Gamma(n+1-ic/2)\Gamma(y+1-ic/2)}{\Gamma(y+n+1-ic/2)\Gamma(1-ic/2)}=1+\frac{-ih_1\Gamma(2+1-ic/2)}{\Gamma(1-ic/2)(y+1-ic/2)}+$$

$$+\frac{(-i)^2h_2\Gamma(3+1-ic/2)}{\Gamma(1-ic/2)(y+1-ic/2)(y+2-ic/2)}+\ldots\ldots ,$$

the series being convergent for Rey > 0. The equation also shows that:

$$g_n^{(3)} = (-i)^n h_n\,\frac{\Gamma(n+1-ic/2)}{\Gamma(1-ic/2)} \quad . \tag{2-105}$$

Substituting (2-105) in (2-97) we obtain:

$$\sum_{m=o}^{3} T_m(n)g_{n-m}^{(3)} = 0 \quad ; \quad g_1^{(3)} = 1 \;;\; g_{-j}^{(3)} = 0 \;,\; j=1,2,3,\ldots. \;, \tag{2-106}$$

where:

$$T_o(n)= \frac{F_o(n)}{(n-ic/2)(n-1-ic/2)(n-2-ic/2)} = 2in \qquad (2\text{-}107)$$

$$T_1(n)=-i\,\frac{F_1(n)}{(n-1-ic/2)(n-2-ic/2)} = -i(n-1-ic/2)[n-\tfrac{1}{2}(5a+3b)]+$$
$$+c(1-ia)+iv(v+1) \qquad (2\text{-}108)$$

$$T_2(n)=-\frac{F_2(n)}{n-2-ic/2}=(a+b)(n-2-ic/2)^2+2b(1-ia)(n-2-ic/2)-$$
$$-(a+b)v(v+1) \qquad (2\text{-}109)$$

$$T_3(n)=iF_3(n)=iab(n-3-v-ic/2)(n-2+v-ic/2) =$$
$$=iab[(n-2-ic/2)(n-3-ic/2)-v(v+1)] \quad . \qquad (2\text{-}110)$$

Next, we are going to prove that, when a, b are real:

$$g_n^{(4)}=\bar{g}_n^{(3)} \quad , \qquad (2\text{-}111)$$

so that, looking at (2-56),(2-57) we should also have:

$$\Omega^{(4)}_{(y)} = \overline{\Omega^{(3)}_{(\bar{y})}} \quad . \qquad (2\text{-}112)$$

Furthermore, since $\bar{\Gamma}(\bar{y}+1-ic/2)=\Gamma(y+1+ic/2)$ and $\overline{i^{\bar{y}}} = \overline{e^{i\pi(y_r-iy_i)/2}} = e^{-i\pi(y_r+iy_i)/2} = (-i)^y$, looking at (2-54) and (2-55) we see that also we have:

$$v_4(y) = \bar{v}_3(\bar{y}) \quad . \qquad (2\text{-}113)$$

This relation holds not only for Re$y>0$ but also for any y, because in (2-50), which provides the analytic continuation of $v_3(y)$ and $v_4(y)$ in the plane Re$y<0$, the coefficients $p_n(y)$ (n=0, 1,2,3,4) are polynomials in y with real coefficients.

In order to prove (2-111) we look at the differential equa-

tion (2-42) satisfied by $\Psi_3(t)$ around $t_3=i$ and $\Psi_4(t)$ around $t_4=-i$. It is an equation with real polynomial coefficients and its two singular points $t_3=i$ and $t_4=-i$ are complex conjugates with exponents $\beta_3=-ic/2=\bar{\beta}_4$. It is then easy to see that $\Psi_4(t)=\bar{\Psi}_3(\bar{t})$, which means that $f_n=\bar{h}_n$ [see (2-45) and (2-46)]. Following the steps leading to (2-105) for $\Omega^{(4)}_{(y)}$ we would find $g_n^{(4)}=i^n f_n \cdot \dfrac{\Gamma(n+1+ic/2)}{\Gamma(1+ic/2)}$ so that, ~~by comparison with (2-105), relation~~ (2-111) follows at once.

A faster but not as rigorous way, is to observe that (2-112) and (2-113) hold at least for $y \xrightarrow[\mathrm{rhp}]{} \infty$, as equations (2-54) and (2-57) show; since the difference equation (2-5) has real polynomial coefficients, the relations should hold for all y.

Turning to (2-106) and (2-110) we can find: $g_0^{(3)}=1$ and

$$g_1^{(3)}=-\frac{T_1(1)}{T_0(1)}=-\frac{(-c/2)[1-i(5a+3b)/2]+c-iac+iv(v+1)}{2i}\ .$$

Thus, the corresponding series expansion in (2-88) can be written:

$$Ce^{-\pi c/4+iz}z^{ic/2}\Big[1+\frac{ic(a+3b)+i4v(v+1)+2c}{8}\,\frac{1}{z}+\sum_{n=2}^{\infty}\frac{g_n^{(3)}}{(iz)^n}\Big]\ . \qquad (2\text{-}114)$$

Equation (1-94) defines $R_3(z)$ as follows:

$$R_3(z)\sim e^{iz}z^{ic/2}\Big[1+\Big(\frac{c}{2}+\frac{(c/2i)(c/2i+1)-v(v+1)-cb}{2i}\Big)\frac{1}{z}+\dots\dots=$$

$$=e^{iz}\ z^{ic/2}\Big[1+\frac{ic(a+3b)+i4v(v+1)+2c}{8}\,\frac{1}{z}+\dots\dots\Big]\ ,$$

that is, the expression (2-114) is simply: $Ce^{-\pi c/4}\,R_3(z)$. An analogous statement holds for $R_4(z)$; we finally conclude that (2-88) can be written:

$$R_{\substack{1\\2}}(z) \underset{|z|\to\infty}{\sim} A_4\, R_4(z) \quad , \; 0<\phi<\pi$$

$$\sim A_3\, R_3(z) \quad , \; -\pi<\phi<0$$

$$\sim A_3\, R_3(z) + A_4\, R_4(z) \quad , \; \phi = 0 \quad , \tag{2-115}$$

where:

$$A_4 = D\, e^{-\pi c/4} \quad , \quad A_3 = C\, e^{-\pi c/4} \quad . \tag{2-116}$$

For $R_1(z)$ we use $\sigma=v+1$ throughout, for $R_2(z)$ (when $2v+1$ is not an integer) $\sigma=-v$ throughout.

Since for real a, b, z, $R_{\substack{1\\2}}(z)$ are real and $R_3(z)=\bar{R}_4(z)$, we must have $A_3=\bar{A}_4$, or from (2-116):

$$C = \bar{D} \quad ; \quad \text{for real } a,\, b \quad , \tag{2-117}$$

a fact that will also be verified in the process of determining these coefficients.

DETERMINATION OF THE COEFFICIENTS OF THE LINEAR RELATIONS

In view of (2-116), our problem is the evaluation of the coefficients C and D of (2-69). Referring to (2-70)-(2-73) we have:

$$Av_1(-3+\sigma)+Bv_2(-3+\sigma)+Cv_3(-3+\sigma)+Dv_4(-3+\sigma) = 0$$

$$Av_1(-2+\sigma)+Bv_2(-2+\sigma)+Cv_3(-2+\sigma)+Dv_4(-2+\sigma) = 0$$

$$Av_1(-1+\sigma)+Bv_2(-1+\sigma)+Cv_3(-1+\sigma)+Dv_4(-1+\sigma) = 0$$

$$Av_1(\sigma)+Bv_2(\sigma)+Cv_3(\sigma)+Dv_4(\sigma) = 1 \quad .$$

Solving for A, B, C, D we obtain:

$$A=\mu^{(4)}_{1(\sigma-3)},\quad B=\mu^{(4)}_{2(\sigma-3)},\quad C=\mu^{(4)}_{3(\sigma-3)},\quad D=\mu^{(4)}_{4(\sigma-3)} \quad , \tag{2-118}$$

where $\mu^{(4)}_{s(y)}$ $(s=1,2,3,4)$ are the cofactors of the last row of the determinant $P(y)$ divided by $P(y)$ and where:

$$P(y) = \begin{vmatrix} v_1(y) & v_2(y) & v_3(y) & v_4(y) \\ v_1(y+1) & v_2(y+1) & v_3(y+1) & v_4(y+1) \\ v_1(y+2) & v_2(y+2) & v_3(y+2) & v_4(y+2) \\ v_1(y+3) & v_2(y+3) & v_3(y+3) & v_4(y+3) \end{vmatrix} \quad . \tag{2-119}$$

$P(y)$ is the Casorati's determinant (17 Chapt. XII, 18 Chapt. I) for the four particular solutions $v_s(y)$ $(s=1,2,3,4)$ of (2-5). It satisfies the following difference equation or the first order (Heymann's theorem):

$$\frac{P(y+1)}{P(y)} = \frac{p_0(y)}{p_4(y)} = \frac{1}{ab(y+3-v)(y+4+v)} \quad , \tag{2-120}$$

whose solution is given by (17 pp. 327-328):

$$P(y) = (1/ab)^y \frac{\bar{\bar{w}}(y)}{\Gamma(y+3-v)\Gamma(y+4+v)} \quad , \tag{2-121}$$

where $\bar{\bar{w}}(y)$ is, in general, an arbitrary periodic function of y with period 1. It can be verified immediately that (2-121) satisfies (2-120). Since we know how $v_s(y)$ $(s=1,2,3,4)$ behave as $y \xrightarrow[\text{rhp}]{} \infty$ we can find the form of $\bar{\bar{w}}(y)$ corresponding to the particular set formed by these four solutions. It turns out to be a constant. As $y \xrightarrow[\text{rhp}]{} \infty$ we use the expressions (2-54)-(2-61) into (2-119) to obtain:

$$P(y) \underset{y \xrightarrow[\text{rhp}]{} \infty}{\sim} (1/ab)^y \frac{1}{y^4 \Gamma(y+1-ic/2)\Gamma(y+1+ic/2)} \quad .$$

$$\cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1/a & -1/b & i/(y+1-ic/2) & -i/(y+1+ic/2) \\ (-1/a)^2 & (-1/b)^2 & 0 & 0 \\ (-1/a)^3 & (-1/b)^3 & 0 & 0 \end{vmatrix}$$

In the determinant subtract the last column from the third and use the difference as the new third column, without changing the value of the determinant. The third column becomes:

$$\begin{matrix} 0 \\ \dfrac{2i(y+1)}{(y+1-ic/2)(y+1+ic/2)} \\ 0 \\ 0 \end{matrix}$$

so that:

$$P(y) \underset{y \xrightarrow[rhp]{} \infty}{\sim} (1/ab)^y \frac{2i}{y^3 \Gamma(y+2-ic/2)\Gamma(y+2+ic/2)} \cdot$$

$$\cdot \begin{vmatrix} 1 & 1 & 0 & 1 \\ -1/a & -1/b & 1 & -i/(y+1+ic/2) \\ (-1/a)^2 & (-1/b)^2 & 0 & 0 \\ (-1/a)^3 & (-1/b)^3 & 0 & 0 \end{vmatrix} =$$

$$=(1/ab)^{y+3} \frac{2ic}{y^3} \frac{1}{\Gamma(y+3-v)\Gamma(y+4+v)} \frac{\Gamma(y+3-v)\Gamma(y+4+v)}{\Gamma(y+2-ic/2)\Gamma(y+2+ic/2)} \cdot$$

But (7 Chapt. VIII, 17 pp. 254-255):

$$\frac{\Gamma(y+\tau)}{\Gamma(y+\rho)} \xrightarrow[y\to\infty]{} y^{\tau-\rho} \tag{2-122}$$

and this holds as $y \to \infty$ in the sector $-\pi+\epsilon < \arg y < \pi-\epsilon$. Applying this to the last expression, we obtain:

$$\frac{1}{y^3} \frac{\Gamma(y+3-v)\Gamma(y+4+v)}{\Gamma(y+2-ic/2)\Gamma(y+2+ic/2)} \xrightarrow[y_{rhp}\infty]{} y^0 = 1 \quad ,$$

so that:

$$P(y) \xrightarrow[y_{\text{rhp}}\to\infty]{} (1/ab)^y \frac{21c}{(ab)^3} \frac{1}{\Gamma(y+3-v)\Gamma(y+4+v)}$$

Comparing with (2-121) we see that $\bar{w}(y) = \dfrac{21c}{(ab)^3}$, a constant.
So finally:

$$P(y) = (1/ab)^{y+3} \frac{21c}{\Gamma(y+3-v)\Gamma(y+4+v)} \quad . \tag{2-123}$$

We next define the so-called multipliers (17 pp. 372-374) $N_s(y)$ $(s=1,2,3,4)$ of the solutions $v_s(y)$ of (2-5) by:

$$N_s(y) = \mu^{(4)}_{s(y+1)}/p_4(y) \ .$$

Here $p_0(y) = 1$ and from (2-120) we obtain: $1/p_4(y)=P(y+1)/P(y)$; therefore:

$$N_s(y) = \mu^{(4)}_{s(y+1)} \ P(y+1)/P(y) \quad .$$

Remembering the definition of $\mu^{(4)}_{s(y)}$ (immediately after (2-118)), we see that $\mu^{(4)}_{s(y+1)}$ $P(y+1)$ are simply the cofactors of the last row of Casorati's determinant $P(y+1)$. We have explicitly:

$$N_1(y) = -\frac{1}{P(y)} \begin{vmatrix} v_2(y+1) & v_3(y+1) & v_4(y+1) \\ v_2(y+2) & v_3(y+2) & v_4(y+2) \\ v_2(y+3) & v_3(y+3) & v_4(y+3) \end{vmatrix} \tag{2-124}$$

$$N_2(y) = \frac{1}{P(y)} \begin{vmatrix} v_1(y+1) & v_3(y+1) & v_4(y+1) \\ v_1(y+2) & v_3(y+2) & v_4(y+2) \\ v_1(y+3) & v_3(y+3) & v_4(y+3) \end{vmatrix} \tag{2-125}$$

$$N_3(y) = -\frac{1}{P(y)} \begin{vmatrix} v_1(y+1) & v_2(y+1) & v_4(y+1) \\ v_1(y+2) & v_2(y+2) & v_4(y+2) \\ v_1(y+3) & v_2(y+3) & v_4(y+3) \end{vmatrix} \tag{2-126}$$

$$N_4(y) = \frac{1}{P(y)} \begin{vmatrix} v_1(y+1) & v_2(y+1) & v_3(y+1) \\ v_1(y+2) & v_2(y+2) & v_3(y+2) \\ v_1(y+3) & v_2(y+3) & v_3(y+3) \end{vmatrix} . \tag{2-127}$$

In the present case, where $p_o(y)=1$, $N_s(y)$ $(s=1,2,3,4)$ are simply the negatives of the cofactors of the first row of $P(y)$ divided by $P(y)$. The original definition is:

$$\mu_{s(y)}^{(4)} = p_4(y-1)N_s(y-1) = ab(y+2-v)(y+3+v)N_s(y-1) . \tag{2-128}$$

Using (2-118) we can express A, B, C, D as follows:

$$A = [ab(y+3-v)(y+4+v)N_1(y)]_{y=-4+\sigma} \tag{2-129}$$

$$B = [ab(y+3-v)(y+4+v)N_2(y)]_{y=-4+\sigma} \tag{2-130}$$

$$C = [ab(y+3-v)(y+4+v)N_3(y)]_{y=-4+\sigma} \tag{2-131}$$

$$D = [ab(y+3-v)(y+4+v)N_4(y)]_{y=-4+\sigma} \tag{2-132}$$

For the expansion of $R_1(z)$ we use $\sigma=v+1$, for that of $R_2(z)$, when $2v+1$ is not an integer, we use $\sigma=-v$.

Working as before for $P(y)$ we can find how $N_s(y)$ $(s=1,2,3,4)$ behave as $y \underset{rhp}{\longrightarrow} \infty$. Using (2-54)-(2-61), (2-122) and (2-123) we find from (2-124):

$$N_1(y) \underset{y \underset{rhp}{\rightarrow} \infty}{\sim} - \frac{\Gamma(y+3-v)\Gamma(y+4+v)(ab)^{y+3}(-1/b)^{y+1}}{2icy\,\Gamma(y+2-ic/2)\Gamma(y+2+ic/2)} \cdot$$

$$\cdot \begin{vmatrix} 1 & 1 & 1 \\ -1/b & i/(y+2-ic/2) & -i/(y+2+ic/2) \\ (-1/b)^2 & 0 & 0 \end{vmatrix} ,$$

or, subtracting the last column in the determinant from the second and using it as the new second column:

$$N_1(y) \underset{y \xrightarrow[rhp]{} \infty}{\sim} - \frac{\Gamma(y+3-v)\Gamma(y+4+v)(ab)^{y+3}(-1/b)^{y+1}}{c\Gamma(y+3-ic/2)\Gamma(y+3+ic/2)} \cdot$$

$$\cdot \begin{vmatrix} 1 & 0 & 1 \\ -1/b & 1 & -i/(y+2+ic/2) \\ 1/b^2 & 0 & 0 \end{vmatrix} ,$$

or:

$$N_1(y) \underset{y \xrightarrow[rhp]{} \infty}{\sim} \frac{(-a)^{y+3}}{c} y \quad . \tag{2-133}$$

Similarly starting from (2-125) we find that:

$$N_2(y) \underset{y \xrightarrow[rhp]{} \infty}{\sim} - \frac{(-b)^{y+3}}{c} \frac{1}{y} \quad . \tag{2-134}$$

Starting from (2-126) we obtain:

$$N_3(y) \underset{y \xrightarrow[rhp]{} \infty}{\sim} = \frac{\Gamma(y+3-v)\Gamma(y+4+v)(ab)^{y+3}(-i/ab)^{y+1}}{2icy^4\Gamma(y+2+ic/2)} \begin{vmatrix} 1 & 1 & 1 \\ -1/a & -1/b & 0 \\ 1/a^2 & 1/b^2 & 0 \end{vmatrix}$$

and by application of (2-122) we can find the following two, asymptotically equivalent, expressions:

$$N_3(y) \underset{y \xrightarrow[rhp]{} \infty}{\sim} - \frac{(-i)^y}{2} \Gamma(y+3-v) y^{v-2-ic/2} \sim$$

$$\sim - \frac{(-i)^y}{2} \Gamma(y+4+v) y^{-v-3-ic/2} \quad . \tag{2-135}$$

Similarly:

$$N_4(y) \underset{y \xrightarrow[rhp]{} \infty}{\sim} - \frac{i^y}{2} \Gamma(y+3-v) y^{v-2+ic/2} \sim - \frac{i^y}{2} \Gamma(y+4+v) y^{-v-3+ic/2} . \tag{2-136}$$

<u>The Adjoint Difference Equation:</u> It is a well-known fact that the multipliers $N_s(y)$ $(s=1,2,3,4)$ are independent solutions of the difference equation adjoint to (2-5), i.e. they satisfy (17 pp. 372-374, 18 Chapt. I):

$$\sum_{m=0}^{4} p_m(y+4-m)N(y+4-m) = 0 \quad ,$$

or:

$$q_4(y)N(y+4)+q_3(y)N(y+3)+q_2(y)N(y+2)+q_1(y)N(y+1)+$$
$$+q_0(y)N(y) = 0 \quad , \tag{2-137}$$

where:

$$q_4(y)=p_0(y+4) = 1 \tag{2-138}$$

$$q_3(y)=p_1(y+3) = 2a \tag{2-139}$$

$$q_2(y)=p_2(y+2)=(y+4)(y+3)+a^2-v(v+1) =$$
$$= (y+2)(y+3)+2(y+2)+a^2+2-v(v+1) \tag{2-140}$$

$$q_1(y)=p_3(y+1)=(a+b)(y+1)(y+2)+(5a+3b)(y+1)+3c+$$
$$+(a+b)[6-v(v+1)] \tag{2-141}$$

$$q_0(y)=p_4(y)=ab(y+3-v)(y+4+v)=ab[y(y+1)+6y+12-v(v+1)] \ . \tag{2-142}$$

Applying the method of Laplace's transformation as before, we form the functions:

$$\phi_2(t) = t^2+(a+b)t+ab = (t+a)(t+b)$$

$$\phi_1(t) = 2t^2+(5a+3b)t+6ab$$

$$\phi_0(t) = t^4+2at^3+[a^2+2-v(v+1)]t^2+[3c+(a+b)(6-v(v+1))]t+ab[12-v(v+1)]$$

The corresponding differential equation for $\Psi(t)$ is:

$$(t+a)(t+b)t^2\Psi''(t)-\phi_1(t)t\Psi'(t)+\phi_0(t)\Psi(t) = 0 \tag{2-143}$$

and has regular singular points at $t=0$, $t=-a$, $t=-b$ and an irregular singularity at $t=\infty$. Indicial equation around $t=0$:

$ab\rho(\rho-1)-6ab\rho+ab[12-v(v+1)] = 0$ with roots $\rho_1 = 4+v$, $\rho_2 = 3-v$. Since v is positive, for $\mathrm{Re}\,y > v-3$ the corresponding $I(\psi,t)\big|_{t=0}$ is zero for any solution of (2-143). The indicial equation around $t=-b$ is:

$\beta(\beta-1)+[\phi_1(-\beta)/(bc)]\beta = 0$ with roots 0 and $\beta_2 = 1-\dfrac{2b^2-(5a+3b)b+6ab}{bc} =$ $= 0$. There exists one solution $\psi_2(t) = 1+d_1(t+\beta)+d_2(t+\beta)^2+\ldots..$ of (2-143) and, correspondingly, a solution of (2-137) in the form:

$$N(y) = \frac{1}{2\pi i}\int_{\ell} t^{y-1}\psi_2(t)\,dt\ , \quad \text{analytic for } \mathrm{Re}\,y > v-3\ .$$

The path ℓ is shown in figure (2-4). As in (2-20), it can easily be shown that $I(\psi_2,t)\big|_{t=-b} = 0$. According to (2-52),(2-53), Nörlund's theorem and the statement on page 2-20, we can expand the above expression, as $y \xrightarrow[\text{rhp}]{} \infty$, as follows:

$$N(y) \underset{y \xrightarrow[\text{rhp}]{} \infty}{\sim} (-b)^y \frac{\Gamma(y)}{\Gamma(y+1)}\,\Omega_{(y)} \sim (-b)^y\,\Omega_{(y)}/y\ .$$

Comparing with (2-134) we see that the above solution is proportional to $N_2(y)$.

Similarly, the other solution, corresponding to the singular point $t=-a$ of (2-143), would be identified as proportional to $N_1(y)$. We are interested, however, only in the solutions $N_3(y)$ and $N_4(y)$ of (2-137), on which C and D depend. The coefficients A, B do not appear in the asymptotic expansions of $R_1(z)$, $R_2(z)$.

We will find $N_3(y)$ and $N_4(y)$ explicitly. We put:

$$N(y) = \Gamma(y+4+v)\,M(y)\ . \qquad (2\text{-}144)$$

Substituting in (2-137) we obtain:

$$Q_4(y)M(y+4)+Q_3(y)M(y+3)+Q_2(y)M(y+2)+Q_1(y)M(y+1)+$$

$$+Q_o(y)M(y) = 0 \quad , \tag{2-145}$$

where, with the use of (2-138)-(2-142), we have:

$$Q_4(y)=q_4(y)\Gamma(y+8+v)/\Gamma(y+5+v)=(y+4)(y+5)(y+6)+3(v+1)(y+4)(y+5)+$$
$$+3(v+1)(v+2)(y+4)+(v+1)(v+2)(v+3) \tag{2-146}$$

$$Q_3(y)=q_3(y)\Gamma(y+7+v)/\Gamma(y+5+v)=2a[(y+3)(y+4)+2(v+2)(y+3)+$$
$$+(v+2)(v+3)] \tag{2-147}$$

$$Q_2(y)=q_2(y)\Gamma(y+6+v)/\Gamma(y+5+v)=(y+2)(y+3)(y+4)+(v+3)(y+2)(y+3)+$$
$$+[a^2+6-v(v-1)](y+2)+(v+3)[a^2+2-v(v+1)] \tag{2-148}$$

$$Q_1(y)=q_1(y)=(a+b)(y+1)(y+2)+(5a+3b)(y+1)+3c+$$
$$+(a+b)[6-v(v+1)] \tag{2-149}$$

$$Q_o(y)=aby+ab(3-v) \quad . \tag{2-150}$$

Applying the method of Laplace's transformation we form the functions:

$$\phi_3(t)=t^4+t^2=t^2(t^2+1) \tag{2-151}$$

$$\phi_2(t)=3(v+1)t^4+2at^3+(v+3)t^2+(a+b)t \tag{2-152}$$

$$\phi_1(t)=3(v+1)(v+2)t^4+4a(v+2)t^3+[a^2+6-v(v-1)]t^2+$$
$$+(5a+3b)t+ab \tag{2-153}$$

$$\phi_o(t)=(v+1)(v+2)(v+3)t^4+2a(v+2)(v+3)t^3+(v+3)[a^2+2-v(v+1)]t^2+$$
$$+[3c+(a+b)(6-v(v+1))]t+ab(3-v) \tag{2-154}$$

The corresponding differential equation for $\Psi(t)$ and expression $I(\Psi,t)$ are:

$$t^2(t^2+1)t^3\psi'''(t)-\phi_2(t)t^2\psi''(t)+\phi_1(t)t\psi'(t)-\phi_o(t)\psi(t)=0 \quad , \qquad (2\text{-}155)$$

$$I(\psi,t)=\psi(t)\sum_{m=o}^{2}\frac{d^m}{dt^m}[t^{y+m}\phi_{m+1}(t)]-\psi'(t)\sum_{m=o}^{1}\frac{d^m}{dt^m}[t^{y+m+1}\phi_{m+2}(t)]+$$
$$+\psi''(t)t^{y+2}\phi_3(t) = \psi(t)t^y[\phi_1(t)+(y+1)\phi_2(t)+t\phi_2'(t)+$$
$$+(y+2)(y+1)\phi_3(t)+2(y+2)t\phi_3'(t)+t^2\phi_3''(t)]-\psi'(t)t^{y+1}[\phi_2(t)+$$
$$+(y+2)\phi_3(t)+t\phi_3'(t)]+\psi''(t)t^{y+2}\phi_3(t) \quad . \qquad (2\text{-}156)$$

Equation (2-155) has three regular singularities at $t_4=i$, $t_3=-i$, $t=\infty$ and an irregular singularity at $t=0$. For its solutions around $t=0$ we put: $\psi(t)=e^{m/t}\sum_{n=o}^{\infty}\alpha_n t^{n+\beta}$, $\alpha_o=1$ and obtain:

$$\psi'(t)=e^{m/t}\sum_{n=o}^{\infty}(n+\beta)\alpha_n t^{n+\beta-1}-me^{m/t}\sum_{n=o}^{\infty}\alpha_n t^{n+\beta-2} \qquad (2\text{-}157)$$

$$\psi''(t)=e^{m/t}\sum_{n=o}^{\infty}(n+\beta)(n+\beta-1)\alpha_n t^{n+\beta-2}-2me^{m/t}\sum_{n=o}^{\infty}(n+\beta-1)\alpha_n t^{n+\beta-3}+$$
$$+m^2e^{m/t}\sum_{n=o}^{\infty}\alpha_n t^{n+\beta-4} \qquad (2\text{-}158)$$

$$\psi'''(t)=e^{m/t}[\sum_{n=o}^{\infty}(n+\beta)(n+\beta-1)(n+\beta-2)\alpha_n t^{n+\beta-3}-3m\sum_{n=o}^{\infty}(n+\beta-1)(n+\beta-2)\cdot$$
$$\cdot\alpha_n t^{n+\beta-4}+3m^2\sum_{n=o}^{\infty}(n+\beta-2)\alpha_n t^{n+\beta-5}-m^3\sum_{n=o}^{\infty}\alpha_n t^{n+\beta-6}] \quad . \qquad (2\text{-}159)$$

Substituting into (2-155) and eliminating the common factor $e^{m/t}t^\beta$ we obtain:

$$(t^2+1)[\sum_{n=o}^{\infty}(n+\beta)(n+\beta-1)(n+\beta-2)\alpha_n t^{n+2}-3m\sum_{n=o}^{\infty}(n+\beta-1)(n+\beta-2)\alpha_n t^{n+1}+$$
$$+3m^2\sum_{n=o}^{\infty}(n+\beta-2)\alpha_n t^n-m^3\sum_{n=o}^{\infty}\alpha_n t^{n-1}]-[3(v+1)t^3+2at^2+(v+3)t+(a+b)]\cdot$$

$$\cdot\left[\sum_{n=0}^{\infty}(n+\beta)(n+\beta-1)\alpha_n t^{n+1}-2m\sum_{n=0}^{\infty}(n+\beta-1)\alpha_n t^n+m^2\sum_{n=0}^{\infty}\alpha_n t^{n-1}\right]+\Big\{3(v+1)\cdot$$

$$\cdot(v+2)t^4+4a(v+2)t^3+[a^2+6-v(v-1)]t^2+(5a+3b)t+ab\Big\}\cdot\left[\sum_{n=0}^{\infty}(n+\beta)\alpha_n t^n-\right.$$

$$\left.-m\sum_{n=0}^{\infty}\alpha_n t^{n-1}\right]-\Big\{(v+1)(v+2)(v+3)t^4+2a(v+2)(v+3)t^3+(v+3)[a^2+2-v(v+1)]\cdot$$

$$\cdot t^2+[3c+(a+b)(6-v(v+1))]t+ab(3-v)\Big\}\sum_{n=0}^{\infty}\alpha_n t^n = 0 \ .$$

The lowest power of t in this expression is t^{-1}. Equating the coefficients of t^{-1} and t^0 to 0 we obtain:

t^{-1}: $-m^3\alpha_0-(a+b)m^2\alpha_0-abm\alpha_0=0$ with roots $m_1=0$, $m_2=-a$, $m_2=-b$.

t^0 : $-m^3\alpha_1+3m^2(\beta-2)\alpha_0-(a+b)m^2\alpha_1-(v+3)m^2\alpha_0+2m(a+b)(\beta-1)\alpha_0-abm\alpha_1-$

$-(5a+3b)m\alpha_0+ab\beta\alpha_0-ab(3-v)\alpha_0 = 0$, or

$(3\beta-v-9)m^2+[2(a+b)(\beta-1)-5a-3b]m+ab(\beta+v-3) = 0$.

For $m_1=0$ we get: $\beta_1= 3-v$.

For $m_2=-a$: $(3\beta_2-v-9)a-2(a+b)(\beta_2-1)+5a+3b+b(\beta_2+v-3)=0$, or $\beta_2= v+2$.

For $m_3=-b$: $(3\beta_3-v-9)b-2(a+b)(\beta_3-1)+5a+3b+a(\beta_3+v-3)=0$, or $\beta_3= v+4$.

Thus, three normal asymptotic solutions around t=0 are obtained:

$$\psi_I(t)\underset{t\to 0}{\sim} t^{3-v}\left[1+\sum_{n=1}^{\infty}\alpha_n^{(I)}t^n\right]$$

$$\psi_{II}(t)\underset{t\to 0}{\sim} e^{-a/t}\ t^{v+2}\left[1+\sum_{n=1}^{\infty}\alpha_n^{(II)}t^n\right]$$

$$\psi_{III}(t)\underset{t\to 0}{\sim} e^{-b/t}\ t^{v+4}\left[1+\sum_{n=1}\alpha_n^{(III)}t^n\right] \ .$$

Upon any ray drawn from t=0, for which $\mathrm{Re}(-a/t)<0$, $\mathrm{Re}(-b/t)<0$ and for any y we have:

$\lim_{t\to 0}t^y\psi_s(t)=\lim_{t\to 0}t^{y+1}\psi_s'(t)=\lim_{t\to 0}t^{y+2}\psi_s''(t) = 0$, for s=II,III.

If we further restrict y to values for which $\mathrm{Re}(y+3-v) > 0$, or $\mathrm{Re}y > v-3$, we are going to have:

$$\lim_{t\to 0} t^y \psi_I(t) = \lim_{t\to 0} t^{y+1}\psi_I'(t) = \lim_{t\to 0} t^{y+2}\psi_I''(t) = 0 \ .$$

Therefore, for any path ℓ in the t-plane, starting and ending at $t=0$ along rays satisfying the conditions $\mathrm{Re}(-a/t) < 0$, $\mathrm{Re}(-b/t) < 0$, while $\mathrm{Re}y > v-3$, we are going to have $I(\psi,t)\big|_\ell = 0$ for any solution $\psi(t)$ of (2-155). This is easily verified by looking at (2-156), (2-151)-(2-154). Furthermore, the same result is still valid, if the path approaches $t=0$ along either of the two limiting rays: $\mathrm{Re}(-a/t)=0$, $\mathrm{Re}(-b/t) \leqq 0$ or $\mathrm{Re}(-a/t) \leqq 0$, $\mathrm{Re}(-b/t)=0$, if y is restricted by $\mathrm{Re}y > \max(v-3,-v-2)$, a condition that reduces to the previous one, $\mathrm{Re}y > v-3$, if $v \geqslant 1/2$. Again, a look at (2-151)-(2-154), (2-156) and (2-157)-(2-159) verifies this statement immediately. As before, these observations will serve to fix the path of integration in the t-plane.

The indicial equation of (2-155) around $t_4=1$ is:

$$\beta(\beta-1)(\beta-2) - \left[\frac{t^2\phi_2(t)}{t^5(t+1)}\right]_{t=1}\beta(\beta-1) = 0$$ with roots 0, 1 and $\beta_4 = v+2-ic/2$.

Indicial equation around $t_3=-1$: $\beta(\beta-1)(\beta-2) - \left[\frac{t^2\phi_2(t)}{t^5(t-1)}\right]_{t=-1}\beta(\beta-1) = 0$ with roots 0, 1 and $\beta_3 = v+2+ic/2$.

There exist two solutions of (2-155) in the form:

$$\psi_4(t-1) = (t-1)^{v+2-ic/2}\left[1+\sum_{n=1}^{\infty} d_n (t-1)^n\right] \ , \quad |t-1|<1 \tag{2-160}$$

$$\psi_3(t+1) = (t+1)^{v+2+ic/2}\left[1+\sum_{n=1}^{\infty} c_n (t+1)^n\right] \ , \quad |t+1|<1 \tag{2-161}$$

and, correspondingly, two solutions for $N(y)$ in the form:

$$N_{(4)}(y) = d_0 \frac{\Gamma(y+4+v)}{2\pi i}\int_{\ell_4} t^{y-1}\psi_4(t-1)\, dt \tag{2-162}$$

$$N_{(3)}(y) = c_o \frac{\Gamma(y+4+v)}{2\pi i} \int_{\ell_3} t^{y-1} \psi_3(t+i)\, dt \quad , \tag{2-163}$$

where the possible paths of integration are shown in figure (2-5). The so defined functions are analytic for $\mathrm{Re}y > v-3$ and ℓ_3, ℓ_4 as in figure (2-5), or for $\mathrm{Re}y > \max(v-3,-v-2)$ and ℓ_3 and/or ℓ_4 tangent to the limiting rays as $t \to 0$. For $v > 1/2$ both cases are equivalent. As $y \underset{rhp}{\longrightarrow} \infty$ we can expand the above functions as before, in the following forms:

$$N_{(4)}(y) \underset{y \underset{rhp}{\to} \infty}{\sim} d_o \frac{\Gamma(y+4+v)\Gamma(y)}{\Gamma(y+v+3-ic/2)} i^y \left[1 + \frac{K_1^{(4)}}{y+v+3-ic/2} + \ldots\ldots\right] \sim$$

$$\sim d_o i^y\, \Gamma(y+4+v) y^{-v-3+ic/2}$$

$$N_{(3)}(y) \underset{y \underset{rhp}{\to} \infty}{\sim} c_o \frac{\Gamma(y+4+v)\Gamma(y)}{\Gamma(y+v+3+ic/2)} (-i)^y \left[1 + \frac{K_1^{(3)}}{y+v+3+ic/2} + \ldots\ldots\right] \sim$$

$$\sim c_o (-i)^y\, \Gamma(y+4+v) y^{-v-3-ic/2} \; .$$

Comparing with (2-135),(2-136) we see that the particular functions $N_4(y)$ and $N_3(y)$ we are looking for, are obtained if we take: $d_o = c_o = -1/2$, i.e.

$$N_4(y) = -\frac{\Gamma(y+4+v)}{2} \frac{1}{2\pi i} \int_{\ell_4} t^{y-1} \psi_4(t-i)\, dt \tag{2-164}$$

$$N_3(y) = -\frac{(y+4+v)}{2} \frac{1}{2\pi i} \int_{\ell_3} t^{y-1} \psi_3(t+i)\, dt \quad . \tag{2-165}$$

They are analytic functions of y for $\mathrm{Re}y > v-3$ and ℓ_3, ℓ_4 as in figure (2-5), or for $\mathrm{Re}y > \max(v-3,-v-2)$ in the limiting case, mentioned above. Their continuation to the left is provided by the difference equation (2-137) itself, i.e. by:

$$N(y) = -\frac{1}{ab(y+3-v)(y+4+v)} [N(y+4) + 2aN(y+3) + q_2(y)N(y+2) +$$

$$+q_1(y)N(y+1)] \quad . \qquad (2\text{-}166)$$

Therefore, they have simple poles at the points:

$$y = v-3-n \quad \text{and} \quad y = -v-4-n \quad , \quad n = 0,1,2,\ldots \quad . \qquad (2\text{-}167)$$

Another remark can be made at this point: When $\mathrm{Re}\beta_3 = v+2-$ $-\mathrm{Im}\,agc/2 > -1$ and/or $\mathrm{Re}\beta_4 = v+2+\mathrm{Im}\,agc/2 > -1$, the integration around the small circles surrounding the points $-i$ and/or i, in ℓ_3 and/or ℓ_4, yields 0, as the radius of the circles is allowed to go to 0. We can then replace the corresponding paths by integrations along the lines ℓ_3' and/or ℓ_4', as shown in figure (2-6). The only change in the expressions (2-164),(2-165) for $N_4(y)$, $N_3(y)$ amounts to an appropriate constant factor in front, easily determined in each case.

Explicit solutions of (2-137) for $N_3(y)$ and $N_4(y)$ can be found by the method used previously to obtain explicitly the solution $v_3(y)$ of the difference equation (2-5). The steps are exactly the same. The factorial series obtained for $v_3(y)$ is convergent for $\mathrm{Re}y > 0$. In the present case the series obtained are at least asymptotic and the process is at least formal. We start from the defining integral representations (2-164),(2-165) and proceed to obtain explicitly the series developments (2-160) and (2-161) for $\Psi_4(t-i)$ and $\Psi_3(t+i)$, which satisfy the differential equation (2-155) with coefficients $\phi_s(t)$ given by (2-151)-(2-154). The final results are as follows:

$$N_{\substack{4\\3}}(y) = -\frac{(\pm i)^y}{2}\,\frac{\Gamma(y+4+v)\Gamma(y)}{\Gamma(y+v+3\mp ic/2)}\,T(y) \qquad (2\text{-}168)$$

$$T(y) = 1+\frac{d_1}{y+v+3\mp ic/2}+\frac{d_2}{(y+v+3\mp ic/2)(y+4+v\mp ic/2)}+\ldots \quad . \qquad (2\text{-}169)$$

Recurrence formula for the coefficients d_n:

$$d_n = \frac{1}{2n}[\tau_1(x)d_{n-1}+\tau_2(x)d_{n-2}+\tau_3(x)d_{n-3}+\tau_4(x)d_{n-4}+\tau_5(x)d_{n-5}+$$
$$+\tau_6(x)d_{n-6}] \quad ; \; d_o = 1 \quad ; \quad d_{-j} = 0 \; , \; j=1,2,3,\ldots.. \qquad (2\text{-}170)$$

$$x = n+v\mp ic/2 \qquad (2\text{-}171)$$

$$\tau_1(x)=11x^2-[14v+17\mp i(7a-3b)]x+4v^2+8v-ac\mp i(4av+3c) \qquad (2\text{-}172)$$

$$\tau_2(x)=-25x^3+[39v+102\pm i(3b-17a)]x^2-[18v^2+81v+89+ab-3a^2\pm i(9b-a(16v+$$
$$+39))]x+2v^3+10v^2+(12-a^2-ab)v-3ac\mp i[(3a+b)v^2+(11a+b)v+3c] \quad (2\text{-}173)$$

$$\tau_3(x)=x\{30x^3-[56v+228\pm i(b-19a)]x^2+[3(11v^2+85v+172-a^2)\pm 2i(3b-$$
$$-a(12v+50))]x-6v^3-65v^2+(2a^2-245)v+9a^2-330\pm i[(7a+b)v^2+(55a+b)v+$$
$$+108a-8b]\} \qquad (2\text{-}174)$$

$$\tau_4(x)=x(x-1)(x-v-5)[-20x^2+(24v+122\mp 110a)x+a^2-7v^2-67v-$$
$$-174\pm 12a(3v+16)] \qquad (2\text{-}175)$$

$$\tau_5(x)=x(x-1)(x-2)(x-v-6)(x-v-5)(7x-4v-25\pm 21a) \qquad (2\text{-}176)$$

$$\tau_6(x)=-x(x-1)(x-2)(x-3)(x-v-7)(x-v-6)(x-v-5) \quad . \qquad (2\text{-}177)$$

Throughout the above relations the upper sign is used for $N_4(y)$, the lower for $N_3(y)$.

Different, but equivalent, expansions for $N_4(y)$, $N_3(y)$ can be found by employing Boole's operational method for solving difference equations with rational coefficients, as explained in reference 17 (Chapt. XIV pp. 434-461). It is essentially the method of undetermined coefficients, completely analogous to the method of Frobenius for differential equations. The method is tedius and lengthy but straightforward. The final results are as follows:

$$N_{\frac{4}{3}}(y) = -\frac{(\pm i)^y}{2}\frac{\Gamma(y+4+v)\Gamma(y+3-v)}{\Gamma(y+6\mp ic/2)}\,T(y) \tag{2-178}$$

$$T(y) = 1+\frac{d_1}{y+6\mp ic/2}+\frac{d_2}{(y+6\mp ic/2)(y+7\mp ic/2)}+\ldots\ldots \;. \tag{2-179}$$

Recurrence formula for the coefficients d_n:

$$d_n = \frac{1}{2n}[d_{n-1}f_4(x)+d_{n-2}f_3(x)+d_{n-3}f_2(x)+d_{n-4}f_1(x)+d_{n-5}f_o(x)] \;;$$

$$d_o = 1 \;;\quad d_{-j} = 0 \;,\; j=1,2,3,\ldots\ldots \tag{2-180}$$

$$x = 2-v-n\pm ic/2 \tag{2-181}$$

$$f_o(x)=x(x-1)(x-2)(x+2v+1)(x+2v)(x+2v-1) \tag{2-182}$$

$$f_1(x)=2(x-1)(x-2)(x+2v)(x+2v-1)(3x+3v-3\mp ia) \tag{2-183}$$

$$f_2(x)=(x-2)(x+2v-1)[14x^2+4(7v-10\mp i2a)x+12v^2-42v+30-a^2\mp$$
$$\mp i4a(2v-3)] \tag{2-184}$$

$$f_3(x)=16x^3+[48v-90\mp i(11a-b)]x^2+2[20v^2-94v+85-a^2\mp i((11a-b)(v-2)+$$
$$+b)]x+8v^3-84v^2+(184-2a^2)v+4a^2-108\mp i[8av^2+(7b-47a)v+$$
$$+44a-8b] \tag{2-185}$$

$$f_4(x)=9x^2+[18v-39\mp i2(3a-b]x+8v^2-40v+42-ac\mp i[2(3a-b)v+$$
$$+7b-15a] \tag{2-186}$$

The sign convention is the same.

The inverse factorial series in all these expressions are at least asymptotic for large $|y|$. It is also obvious that with real a, b: $d_n^{(3)} = \bar{d}_n^{(4)}$. Then: $N_3(y) = \bar{N}_4(\bar{y})$, or for real y: $N_3(y)=\bar{N}_4(y)$. Referring to (2-131),(2-132) and with $y=\sigma-4=v-3$ for $R_1(z)$, $y=-v-4$ for $R_2(z)$ (when 2v+1 is not an integer), both of which are real

values, we conclude that: $C = \bar{\bar{D}}$, a fact which was inferred previously, equation (2-117), through different considerations.

It may also be pointed out that different expansions for $N_4(y)$, $N_3(y)$ can be obtained, which may prove better, from the computational point of view, depending on the particular values of a and b under consideration, if in (2-137) we use the general substitution:

$$N(y) = \Gamma(z+h)\ M(z) \quad , \quad z = y+r \tag{2-187}$$

and solve for $M(z)$, employing either of the above methods. The parameters h and r can be chosen conveniently in each case so as to optimize, from the computational point of view, the expansions obtained. Both expansions, given explicitly above, were used for the computations in Chapter 3, PART I, yielding the same values for C and D. The former proved better in certain cases. Others, based on the change of variable (2-187), were also used for comparison and check on the results.

The evaluation of C and D is based on equations (2-131), (2-132), with $y=\sigma-4=v-3$ for $R_1(z)$, $y=\sigma-4=-v-4$ for $R_2(z)$, when, in the latter case, $2v+1$ is not equal to a positive integer. In general, these values will not be useful for direct evaluation of $N(v-3)$ and $N(-v-4)$. However, we can evaluate $N_4(y)$, $N_3(y)$ at y, y+1, y+2, y+3, where y is adequately large and then use the difference equation (2-166) itself to obtain values for $N(y-1)$, $N(y-2)$ etc., up to $N(v-3)$ and $N(-v-4)$.

All these considerations can be expressed in another more compact and general form, which will also prove necessary in the next section, where the case of integral values for $2v+1$ is investigated.

Instead of the initial conditions (2-70)-(2-73), we make use of the general ones given in (2-68): $\bar{\bar{v}}(n+\sigma) = a_n$ for all integers n. It has been proved that they are equivalent to the four expressed by (2-70)-(2-73). The function $\bar{\bar{v}}(y)$ is defined in (2-69).

Starting from (2-119), the definition of Casorati's determinant, we multiply the first, second, third and fourth columns by A, B, C, D, respectively, add and use the sum as the new fourth column. Applying (2-69) we obtain:

$$P(y) = \frac{1}{D}\begin{vmatrix} v_1(y) & v_2(y) & v_3(y) & \bar{\bar{v}}(y) \\ v_1(y+1) & v_2(y+1) & v_3(y+1) & \bar{\bar{v}}(y+1) \\ v_1(y+2) & v_2(y+2) & v_3(y+2) & \bar{\bar{v}}(y+2) \\ v_1(y+3) & v_2(y+3) & v_3(y+3) & \bar{\bar{v}}(y+3) \end{vmatrix} = \frac{P_4(y)}{D} \quad . \tag{2-188}$$

This, incidentally, shows that the constant D is equal to the constant ratio of two of Casorati's determinants of the difference equation (2-5): $D = P_4(y)/P(y)$, where $P(y)$ corresponds to the four particular and independent solutions $v_s(y)$, $s=1,2,3,4$ of (2-5), while $P_4(y)$ corresponds to the set of solutions $v_s(y)$, $s=1,2,3$ and $\bar{\bar{v}}(y)$ of (2-5). From (2-188) we obtain:

$$D = \frac{\bar{\bar{v}}(y+3)}{P(y)}\begin{vmatrix} v_1(y) & v_2(y) & v_3(y) \\ v_1(y+1) & v_2(y+1) & v_3(y+1) \\ v_1(y+2) & v_2(y+2) & v_3(y+2) \end{vmatrix} - \frac{\bar{\bar{v}}(y+2)}{P(y)} \cdot$$

$$\cdot \begin{vmatrix} v_1(y) & v_2(y) & v_3(y) \\ v_1(y+1) & v_2(y+1) & v_3(y+1) \\ v_1(y+3) & v_2(y+3) & v_3(y+3) \end{vmatrix} + \frac{\bar{\bar{v}}(y+1)}{P(y)}\begin{vmatrix} v_1(y) & v_2(y) & v_3(y) \\ v_1(y+2) & v_2(y+2) & v_3(y+2) \\ v_1(y+3) & v_2(y+3) & v_3(y+3) \end{vmatrix} -$$

$$- \frac{\bar{\bar{v}}(y)}{P(y)}\begin{vmatrix} v_1(y+1) & v_2(y+1) & v_3(y+1) \\ v_1(y+2) & v_2(y+2) & v_3(y+2) \\ v_1(y+3) & v_2(y+3) & v_3(y+3) \end{vmatrix} \quad . \tag{2-189}$$

We call (α), (β), (γ), (δ) the four terms of (2-189). Making use of (2-177) and (2-120), i.e. $P(y+1) = P(y)/p_4(y)$, we have:

$$(\alpha) = [\bar{\bar{v}}(y+3)/P(y)]P(y-1)N_4(y-1) = \bar{\bar{v}}(y+3)p_4(y-1)N_4(y-1) \tag{2-190}$$

$$(\delta) = -\bar{\bar{v}}(y)N_4(y) \quad . \tag{2-191}$$

For (β) we observe that for $s=1,2,3$ we have from (2-5):

$$v_s(y+3) = -[p_3(y-1)v_s(y+2)+p_2(y-1)v_s(y+1)+2av_s(y)+v_s(y-1)]/p_4(y-1).$$

Then we substitute into the last row of the determinant in (β) and break it up into four determinants with the same upper two rows, as in (β), and with last rows containing each of the four terms in the above expansion, respectively. The second and third of these determinants vanish because they have two proportional rows. There remains:

$$(\beta)= \frac{\bar{\bar{v}}(y+2)}{P(y)}\,\frac{p_3(y-1)}{p_4(y-1)}\begin{vmatrix} v_1(y) & v_2(y) & v_3(y) \\ v_1(y+1) & v_2(y+1) & v_3(y+1) \\ v_1(y+2) & v_2(y+2) & v_3(y+2)\end{vmatrix} + \frac{\bar{\bar{v}}(y+2)}{P(y)}\,\frac{1}{p_4(y-1)}\cdot$$

$$\cdot\begin{vmatrix} v_1(y) & v_2(y) & v_3(y) \\ v_1(y+1) & v_2(y+1) & v_3(y+1) \\ v_1(y-1) & v_2(y-1) & v_3(y-1)\end{vmatrix} \quad .$$

Interchanging first and third rows, then second and third, in the last determinant, we finally obtain:

$$(\beta) = \frac{\bar{\bar{v}}(y+2)}{P(y)}\,\frac{p_3(y-1)}{p_4(y-1)}\,P(y-1)N_4(y-1)+ \frac{\bar{\bar{v}}(y+2)}{P(y)}\,\frac{1}{p_4(y-1)}\,P(y-2)N_4(y-2)=$$

$$= \bar{\bar{v}}(y+2)p_3(y-1)N_4(y-1)+\bar{\bar{v}}(y+2)p_4(y-2)N_4(y-2) \quad . \qquad (2\text{-}192)$$

Similarly for (γ) we substitute the elements of the first row using:

$$v_s(y) = -[p_4(y)v_s(y+4)+p_3(y)v_s(y+3)+p_2(y)v_s(y+2)+2av_s(y+1)],$$

with $s=1,2,3$. We finally obtain:

$$(\gamma) = -\frac{\bar{\bar{v}}(y+1)}{P(y)}\,p_4(y)\begin{vmatrix} v_1(y) & v_2(y) & v_3(y) \\ v_1(y+2) & v_2(y+2) & v_3(y+2) \\ v_1(y+3) & v_2(y+3) & v_3(y+3)\end{vmatrix} - \frac{\bar{\bar{v}}(y+1)}{P(y)}\,2a \cdot$$

$$\cdot\begin{vmatrix} v_1(y+1) & v_2(y+1) & v_3(y+1) \\ v_1(y+2) & v_2(y+2) & v_3(y+2) \\ v_1(y+3) & v_2(y+3) & v_3(y+3) \end{vmatrix} = -\frac{\bar{\bar{v}}(y+1)}{P(y)}\, p_4(y)P(y+1)N_4(y+1) -$$

$$-\frac{\bar{\bar{v}}(y+1)}{P(y)}2aP(y)N_4(y) = -\bar{\bar{v}}(y+1)N_4(y+1) - 2a\bar{\bar{v}}(y+1)N_4(y) \quad . \qquad (2\text{-}193)$$

Substituting (2-190)-(2-193) into (2-189), we finally obtain:

$$D = \bar{\bar{v}}(y+2)p_4(y-2)N_4(y-2) + [\bar{\bar{v}}(y+3)p_4(y-1) + \bar{\bar{v}}(y+2)p_3(y-1)]N_4(y-1) -$$

$$-[\bar{\bar{v}}(y) + 2a\bar{\bar{v}}(y+1)]N_4(y) - \bar{\bar{v}}(y+1)N_4(y+1) \quad . \qquad (2\text{-}194)$$

For $y=n+\sigma$ and relation (2-68): $\bar{\bar{v}}(n+\sigma)=a_n$ for all n, we obtain:

$$D = a_{n+2}p_4(n-2+\sigma)N_4(n-2+\sigma) + [a_{n+3}p_4(n-1+\sigma) + a_{n+2}p_3(n-1+\sigma)]N_4(n-1+\sigma) -$$

$$-[a_n + 2a\ a_{n+1}]N_4(n+\sigma) - a_{n+1}N_4(n+1+\sigma) \quad . \qquad (2\text{-}195)$$

For an appropriately large value of n, the four values of $N_4(y)$ appearing in (2-195) can be evaluated. As a check, we evaluate D using (2-195) for 10 or 12 values of n, either consecutive or not. For n=-3 we simply get: $D=p_4(\sigma-4)N_4(\sigma-4)$, as in (2-132). C is given by the same equation (2-195), if $N_3(y)$ is substituted in place of $N_4(y)$.

2v+1 IS EQUAL TO AN INTEGER

Asymptotic Expansion of $R_2(z)$ for Large $|z|$: For the coefficients B_m of the logarithmic solution $R_2(z)$ we obtained in Chapter 1, PART II, the recurrence formula:

$$B_m f_o(m+v+1) + B_{m-1}f_1(m+v) + B_{m-2}f_2(m+v-1) + B_{m-3}f_3(m+v-2) + B_{m-4}f_4(m+v-3) =$$

$$= a_m F_o(m+v+1) + a_{m-1}F_1(m+v) + a_{m-2}F_2(m+v-1) \quad . \qquad (1\text{-}55)$$

The b_n's are connected to the B_m's through:

$$n = 2v+1+m \quad , \quad b_n = b_{2v+1+m} = B_m \quad . \tag{1-54}$$

The initial conditions for these formulas are found from (1-53) and (1-54) to be:

$$B_{-2v-4} = B_{-2v-3} = B_{-2v-2} = 0 \ , \quad B_{-2v-1} = b_o \text{ as in (1-51)} \ . \tag{2-196}$$

With $B_o = b_{2v+1} = 0$ and b_o as in (1-51), these conditions can be replaced by the equivalent set:

$$B_o = 0 \ , \ B_{-1} = b_{2v} = b_o d_{2v} \ , \ B_{-2} = b_{2v-1} = b_o d_{2v-1} \ ,$$
$$B_{-3} = b_{2v-2} = b_o d_{2v-2} \quad , \tag{2-197}$$

where b_o and d_{2v}, d_{2v-1}, d_{2v-2} are definite numbers defined previously, equation (1-50). It is easily checked that (2-197) lead to the values (2-196) through the difference equation (1-55) and the special value of b_o in (1-51).

We next write m+4 for m in (1-55). Remembering from (2-68) and (2-69) that in this case $a_m = \bar{\bar{v}}(m+v+1)$ we obtain:

$$f_o(m+v+5)B_{m+4} + f_1(m+v+4)B_{m+3} + f_2(m+v+3)B_{m+2} + f_3(m+v+2)B_{m+1} +$$
$$+ f_4(m+v+1)B_m = F_o(m+v+5)\bar{\bar{v}}(m+v+5) + F_1(m+v+4)\bar{\bar{v}}(m+v+4) +$$
$$+ F_2(m+v+3)\bar{\bar{v}}(m+v+3) \quad . \tag{2-198}$$

We introduce at this point in place of m+v+1 the general variable y and in place of B_m the function v(y) such that:

$$v(m+v+1) = B_m \text{ for all integral values of m.} \tag{2-199}$$

Equation (2-198) transforms into:

$$\sum_{m=o}^{4} f_m(y+4-m)v(y+4-m) = \sum_{m=o}^{2} F_m(y+4-m)\bar{\bar{v}}(y+4-m) \quad .$$

This equation for $y = m+v+1$, m being an integer, readily reduces to (2-198), if (2-199) is also used. We write it finally as follows:

$$p_4(y)v(y+4)+p_3(y)v(y+3)+p_2(y)v(y+2)+p_1(y)v(y+1)+p_o(y)v(y) =$$

$$= q_2(y)\bar{\bar{v}}(y+4)+q_1(y)\bar{\bar{v}}(y+3)+q_o(y)\bar{\bar{v}}(y+2) \quad , \tag{2-200}$$

where, with the use of (1-38)-(1-45), we have:

$$p_4(y)=f_o(y+4)=ab[(y+4)(y+3)-v(v+1)]=ab(y+3-v)(y+4+v) \tag{2-201}$$

$$p_3(y)=f_1(y+3)=(a+b)(y+3)(y+2)+c(y+3)-(a+b)v(v+1) \tag{2-202}$$

$$p_2(y)=f_2(y+2)=(y+2)(y+1)+a^2-v(v+1) \tag{2-203}$$

$$p_1(y)=f_3(y+1)=2a \tag{2-204}$$

$$p_o(y)=f_4(y)= 1 \tag{2-205}$$

$$q_2(y)=F_o(y+4)= -ab[2(y+4)-1]= -\frac{dp_4(y)}{dy} \tag{2-206}$$

$$q_1(y)=F_1(y+3)=-(a+b)[2(y+3)-1]-c=-2(a+b)(y+3)+2b=$$

$$= -\frac{dp_3(y)}{dy} \tag{2-207}$$

$$q_o(y)=F_2(y+2)=-2(y+2)+1=-2y-3= -\frac{dp_2(y)}{dy} \quad . \tag{2-208}$$

The last expressions for $q_s(y)$ as $-[dp_{s+2}(y)/dy]$, $s=0,1,2$ are found from (2-201)-(2-203) by differentiation. In (2-200) the function $\bar{\bar{v}}(y)$ is the solution of the difference equation (2-5) which corresponds to $R_1(z)$, i.e. the one that can be expressed as:

$$\bar{\bar{v}}(y) = Av_1(y)+Bv_2(y)+Cv_3(y)+Dv_4(y) \tag{2-209}$$

and which reduces for $y=n+v+1$ to $\bar{\bar{v}}(n+v+1)=a_n$, the coefficients of $R_1(z)$; $v_s(y)$, $s=1,2,3,4$, in (2-209) are the previously found

four independent particular solutions of (2-5).

Comparison of $p_s(y)$, $s=0,1,2,3,4$, as given by (2-201)-(2-205), with (2-6)-(2-10) shows that they are identical with the $p_s(y)$ of (2-5). That is, the corresponding to this case difference equation (2-200) is an inhomogeneous difference equation whose homogeneous part is identical with (2-5). We are looking for the special solution $v_i(y)$ of this equation, which satisfies the conditions (2-199), i.e.

$$v_i(n+v+1) = B_n = b_{n+2v+1} \quad \text{for all integers } n \quad . \qquad (2\text{-}210)$$

As before, only four of these conditions are sufficient, i.e. from (2-197):

$$v_i(v+1)=0 \ , \ v_i(v)=B_{-1}=b_o d_{2v} \ , \ v_i(v-1)=B_{-2}=b_o d_{2v-1} \ ,$$
$$v_i(v-2)=B_{-3}=b_o d_{2v-2} \ ;$$

More generally we can use (2-210) for n, n+1, n+2, n+3, where n is any integer and $B_n = b_{n+2v+1}$, the coefficients appearing in the definition (1-57) for $R_2(z)$.

The general solution of the inhomogeneous difference equation (2-200) consists of the general solution of the homogeneous equation, i.e. of (2-5), plus a particular solution $v_5(y)$ of the inhomogeneous equation (17 Chapt. XII). That is, we are going to have:

$$v_i(y) = Ev_1(y)+Fv_2(y)+Gv_3(y)+Hv_4(y)+v_5(y) = V(y)+v_5(y) \ , \qquad (2\text{-}211)$$

where E, F, G, H are constants, which are going to be determined so that the initial conditions (2-210) (just four of them) are satisfied.

In order to obtain a particular solution $v_5(y)$ of (2-200) we consider the function $\bar{v}(y)$ as defined in (2-209). It satisfies equation (2-5), i.e.

$$p_4(y)\bar{\bar{v}}(y+4)+p_3(y)\bar{\bar{v}}(y+3)+p_2(y)\bar{\bar{v}}(y+2)+p_1(y)\bar{\bar{v}}(y+1)+p_o(y)\bar{\bar{v}}(y) = 0 \;.$$

Differentiating with respect to y and remembering from (2-206)-(2-208) that $\frac{dp_4(y)}{dy} =-q_2(y)$, $\frac{dp_3(y)}{dy} =-q_1(y)$, $\frac{dp_2(y)}{dy} =-q_o(y)$, while $\frac{dp_1(y)}{dy} = \frac{dp_o(y)}{dy} = 0$, we obtain:

$$p_4(y)\frac{d\bar{\bar{v}}(y+4)}{dy} +p_3(y)\frac{d\bar{\bar{v}}(y+3)}{dy} +p_2(y)\frac{d\bar{\bar{v}}(y+2)}{dy} +p_1(y)\frac{d\bar{\bar{v}}(y+1)}{dy} + +p_o(y)\frac{d\bar{\bar{v}}(y)}{dy} = q_2(y)\bar{\bar{v}}(y+4)+q_1(y)\bar{\bar{v}}(y+3)+q_o(y)\bar{\bar{v}}(y+2) \quad . \qquad (2\text{-}212)$$

After comparison with (2-200) this equation shows that a particular solution of (2-200) is:

$$v_5(y)= \frac{d\bar{\bar{v}}(y)}{dy} = A\frac{dv_1(y)}{dy} +B\frac{dv_2(y)}{dy} +C\frac{dv_3(y)}{dy} +D \frac{dv_4(y)}{dy} \;, \qquad (2\text{-}213)$$

where A, B, C, D are the definite constants corresponding to the first solution $R_1(z)$ with $\sigma=v+1$. A complete solution of (2-200) has thus been found.

In order to obtain the asymptotic expansion of $R_2(z)$ for large $|z|$ we proceed as before, for $R_1(z)$, making use of Theorems I and VI. From its definition in (1-57) $R_2(z)$ can be written:

$$R_2(z)=(\ln z)\,z^{v+1}\sum_{n=o}^{\infty}\bar{\bar{v}}(n+v+1)\,z^n+z^{-v}\sum_{n=o}^{2v}b_n z^n+z^{-v}\sum_{n=2v+1}^{\infty}b_n z^n \;.$$

In the last summation put $n=2v+1+m$; then from (2-210): $b_{2v+1+m}= = v_1(m+v+1)$, so that for $|z|<\min(|a|,|b|)$ we have:

$$R_2(z)=z^{-v}\sum_{n=o}^{2v}b_n z^n+(\ln z)\,z^{v+1}\sum_{n=o}^{\infty}\bar{\bar{v}}(n+v+1)\,z^n+z^{v+1}\sum_{m=o}^{\infty}v_1(m+v+1)\,z^m=$$

$$=z^{-v}\sum_{n=o}^{2v}b_n z^n+(\ln z)\,z^{v+1}\sum_{n=o}^{\infty}\bar{\bar{v}}(n+v+1)\,z^n+z^{v+1}\sum_{n=o}^{\infty}v_5(n+v+1)\,z^n+$$

$$+z^{n+1}\sum_{n=o}^{\infty}[Ev_1(n+v+1)+Fv_2(n+v+1)+Gv_3(n+v+1)+Hv_4(n+v+1)]z^n \ ,$$

where the index n replaced m in the last summation and use of (2-211) was made. Thus we can write:

$$R_2(z)=z^{-v}\sum_{n=o}^{2v}b_n z^n+u_1(z)+u_2(z) \ , \quad |z|<\min(|a|,|b|) \ , \tag{2-214}$$

where:

$$u_1(z)=(\ln z)z^{v+1}\sum_{n=o}^{\infty}\bar{v}(n+v+1)z^n+z^{v+1}\sum_{n=o}^{\infty}v_5(n+v+1)z^n \tag{2-215}$$

$$u_2(z)=z^{v+1}\sum_{n=o}^{\infty}V(n+v+1)z^n=z^{v+1}\sum_{n=o}^{\infty}[Ev_1(n+v+1)+Fv_2(n+v+1)+$$

$$+Gv_3(n+v+1)+Hv_4(n+v+1)]z^n \ . \tag{2-216}$$

Take $u_2(z)$ first. For large $|z|$ its asymptotic expansion is given as (2-87) shows for $R_1(z)$ and for $\sigma=v+1$. It was also shown, immediately after equation (2-114), that:

$$e^{iz}z^{ic/2}[1+\sum_{n=1}^{\infty}g_n^{(3)}/(iz)^n]=R_3(z) \quad \text{and} \quad e^{-iz}z^{-ic/2}[1+\sum_{n=1}^{\infty}g_n^{(4)}/(-iz)^n]=$$

$= R_4(z)$. Thus, as in (2-87):

$$u_2(z)\sim -z^{v+1}\sum_{n=1}^{\infty}\frac{V(-n+v+1)}{z^n}+He^{-\pi c/4}R_4(z) \ , \ 0<\phi<\pi$$

$$\sim -z^{v+1}\sum_{n=1}^{\infty}\frac{V(-n+v+1)}{z^n}+Ge^{-\pi c/4}R_3(z) \ , \ -\pi<\phi<0$$

$$\sim -z^{v+1}\sum_{n=1}^{\infty}\frac{V(-n+v+1)}{z^n}+e^{-\pi c/4}[GR_3(z)+HR_4(z)] \ , \ \phi=0 \ . \tag{2-217}$$

Next take $u_1(z)$. It can be written as follows:

$$u_1(z)=[\frac{\partial}{\partial\sigma}(z^{\sigma}\sum_{n=o}^{\infty}\bar{v}(n+\sigma)z^n)]_{\sigma=v+1} = [\frac{\partial}{\partial\sigma}u_3(z,\sigma)]_{\sigma=v+1} \ , \tag{2-218}$$

where:

$$u_3(z,\sigma) = z^{\sigma}\sum_{n=o}^{\infty}\bar{\bar{v}}(n+\sigma)z^n \ , \quad |z|<\min(|a|,|b|) \quad . \tag{2-219}$$

In fact, performing the differentiation we have:

$$u_1(z)=[z^{\sigma}(\ln z)\sum_{n=o}^{\infty}\bar{\bar{v}}(n+\sigma)z^n+ z^{\sigma}\sum_{n=o}^{\infty}\frac{\partial\bar{\bar{v}}(n+\sigma)}{\partial\sigma}z^n]_{\sigma=v+1} =$$

$$= (\ln z)z^{v+1}\sum_{n=o}^{\infty}\bar{\bar{v}}(n+v+1)z^n + z^{v+1}\sum_{n=o}^{\infty}[\frac{\partial\bar{\bar{v}}(n+\sigma)}{\partial\sigma}]_{\sigma=v+1}\ z^n \ .$$

But:

$$[\frac{\partial\bar{\bar{v}}(n+\sigma)}{\partial\sigma}]_{\sigma=v+1}=[\frac{d\bar{\bar{v}}(n+\sigma)}{d(n+\sigma)}]_{n+\sigma=n+v+1}=[v_5(n+\sigma)]_{n+\sigma=n+v+1}= v_5(n+v+1),$$

where use of the definition (2-213) was also made. Thus:

$$u_1(z) = (\ln z)z^{v+1}\sum_{n=o}^{\infty}\bar{\bar{v}}(n+v+1)z^n+ z^{v+1}\sum_{n=o}^{\infty}v_5(n+v+1)z^n$$

in exact agreement with (2-215). The part $u_1(z)$ in $R_2(z)$, in the form shown by (2-218), could be written down from the beginning, if, in order to obtain the second solution $R_2(z)$ when $2v+1$ is an integer, we had followed the general method of Frobenius, as applied in this special case and explained in reference 9 (pp. 396-404).

The asymptotic expansion for large $|z|$ of $u_3(z,\sigma)$, as given in (2-219), can also be written down following (2-87), with σ considered now as a variable parameter. Referring also to (2-209) we have:

$$u_3(z,\sigma) \underset{|z|\to\infty}{\sim} -z^{\sigma}\sum_{n=1}^{\infty}\frac{\bar{\bar{v}}(-n+\sigma)}{z^n} + De^{-\pi c/4}\ R_4(z) \ , \quad 0<\phi<\pi$$

$$-z^{\sigma}\sum_{n=1}^{\infty}\frac{\bar{\bar{v}}(-n+\sigma)}{z^n} + Ce^{-\pi c/4}\ R_3(z) \ , \quad -\pi<\phi<0$$

$$-z^{\sigma}\sum_{n=1}^{\infty}\frac{\bar{\bar{v}}(-n+\sigma)}{z^n} +e^{-\pi c/4}[CR_3(z)+DR_4(z)] \ , \quad \phi=0 \ . \tag{2-220}$$

Here σ is a variable parameter varying around the root $v+1$ of the indicial equation about $z=0$ of equation (I 1-50). $u_3(z,\sigma)$, as given in (2-219) for $|z|<\min(|a|,|b|)$, is a uniformly convergent series of analytic functions of σ ($\bar{\bar{v}}(y)$ was proved to be analytic for all y and z^σ is an analytic function of σ) and can, therefore, be differentiated any number of times with respect to σ (9 p. 400). Then, for any σ, we let $|z|\to\infty$ and obtained the asymptotic expansions (2-220) for $u_3(z,\sigma)$. Observe that the dependence of $u_3(z,\sigma)$ on σ appears now (and for z in any sector) only in the series $-z^\sigma\sum_{n=1}^{\infty}\bar{\bar{v}}(-n+\sigma)/z^n$. $\bar{\bar{v}}(y)$ is analytic for all y and z^σ is analytic for all σ. For $\sigma=v+1$ we have: $\bar{\bar{v}}(-n+v+1) = a_{-n} = 0$ ($n=1,2,3,\ldots\ldots$). Then, with σ varying in the vicinity of $v+1$ and for sufficiently large $|z|$, $-z^\sigma\sum_{n=1}^{\infty}\bar{\bar{v}}(-n+\sigma)/z^n$ is a uniformly convergent series of analytic functions of σ. Therefore, differentiation with respect to σ any number of times is permissible, as before. Thus we obtain:

$$\frac{\partial u_3(z,\sigma)}{\partial\sigma}\underset{|z|\to\infty}{\sim} -\frac{\partial}{\partial\sigma}\left[z^\sigma\sum_{n=1}^{\infty}\frac{\bar{\bar{v}}(-n+\sigma)}{z^n}\right]=-(\ln z)\,z^\sigma\sum_{n=1}^{\infty}\frac{\bar{\bar{v}}(-n+\sigma)}{z^n} -$$

$$-z^\sigma\sum_{n=1}^{\infty}\frac{\partial\bar{\bar{v}}(-n+\sigma)}{\partial\sigma}\,\frac{1}{z^n}\quad,\quad -\pi<\phi<\pi\ . \tag{2-221}$$

Finally for $\sigma=v+1$:

$$u_1(z)=\left[\frac{\partial}{\partial\sigma}u_3(z,\sigma)\right]_{\sigma=v+1}\underset{|z|\to\infty}{\sim}-(\ln z)\,z^{v+1}\sum_{n=1}^{\infty}\frac{\bar{\bar{v}}(-n+v+1)}{z^n} -$$

$$-z^{v+1}\sum_{n=1}^{\infty}\left[\frac{\partial\bar{\bar{v}}(-n+\sigma)}{\partial(-n+\sigma)}\right]_{\sigma=v+1}(1/z^n)\quad,\quad -\pi<\phi<\pi\ .$$

But: $\bar{\bar{v}}(-n+v+1)=a_{-n}=0$ ($n=1,2,3,\ldots.$) and $\left[\frac{\partial\bar{\bar{v}}(-n+\sigma)}{\partial(-n+\sigma)}\right]_{\sigma=v+1} =$

$= v_5(-n+v+1)$. Therefore:

$$u_1(z) \underset{|z|\to\infty}{\sim} -z^{v+1}\sum_{n=1}^{\infty} v_5(-n+v+1)\,\frac{1}{z^n} \quad . \tag{2-222}$$

Substituting (2-217) and (2-222) in (2-214) we obtain:

$$R_2(z) \underset{|z|\to\infty}{\sim} z^{-v}\sum_{n=0}^{2v} b_n z^n - z^{v+1}\sum_{n=1}^{\infty} v_5(-n+v+1)\frac{1}{z^n} - z^{v+1}\sum_{n=1}^{\infty}\frac{V(-n+v+1)}{z^n} +$$

$$+He^{-\pi c/4}\, R_4(z) \quad , \quad 0<\phi<\pi$$

$$\sim z^{-v}\sum_{n=0}^{2v} b_n z^n - z^{v+1}\sum_{n=1}^{\infty} v_5(-n+v+1)\frac{1}{z^n} - z^{v+1}\sum_{n=1}^{\infty}\frac{V(-n+v+1)}{z^n} +$$

$$+Ge^{-\pi c/4}\, R_3(z) \quad , \quad -\pi<\phi<0$$

$$\sim z^{-v}\sum_{n=0}^{2v} b_n z^n - z^{v+1}\sum_{n=1}^{\infty} v_5(-n+v+1)\frac{1}{z^n} - z^{v+1}\sum_{n=1}^{\infty}\frac{V(-n+v+1)}{z^n} +$$

$$+e^{-\pi c/4}[GR_3(z)+HR_4(z)] \quad , \quad \phi=0 \quad .$$

According to (2-211) and (2-210): $v_5(-n+v+1)+V(-n+v+1)=v_1(-n+v+1)=$ $=b_{-n+2v+1}$. Thus, the first three summations in the above expansions can be combined as follows:

$$z^{-v}\sum_{n=0}^{2v} b_n z^n - z^{v+1}\sum_{n=1}^{\infty}\frac{b_{-n+2v+1}}{z^n} = z^{-v}\sum_{n=0}^{2v} b_n z^n - z^{v+1}\sum_{n=1}^{2v+1}\frac{b_{-n+2v+1}}{z^n} \,,$$

since $b_{-1}=b_{-2}=\ldots\ldots=0$. In the last summation change the index as follows: $n = 2v+1-m$. The expression becomes:

$$z^{-v}\sum_{n=0}^{2v} b_n z^n - z^{v+1}\sum_{m=2v}^{m=0}\frac{b_m}{z^{2v+1-m}} = z^{-v}\sum_{n=0}^{2v} b_n z^n - z^{-v}\sum_{n=2v}^{0}\frac{b_n}{z^{-n}} =$$

$$= z^{-v}\sum_{n=0}^{2v} b_n z^n - z^{-v}\sum_{n=0}^{2v} b_n z^n = 0 \,,$$ i.e. it vanishes. Therefore:

$$R_2(z) \underset{|z|\to\infty}{\sim} A_{24}\, R_4(z) \quad , \quad 0<\phi<\pi$$

$$\sim A_{23}\, R_3(z) \quad , \quad -\pi<\phi<0$$

$$\sim A_{23}\, R_3(z) + A_{24}\, R_4(z) \quad , \quad \phi=0 \quad , \tag{2-223}$$

where:

$$A_{24} = He^{-\pi c/4} \quad , \quad A_{23} = Ge^{-\pi c/4} \quad . \tag{2-224}$$

The result is in complete agreement with theory (9 pp. 168-174, 10 pp. 72-73) and analogous to the result for $R_1(z)$.

Determination of the Coefficients of the Linear Relations: In order to evaluate G and H we follow a method similar to the one which led to expression (2-194) for D. From (2-111) and (2-213) we have:

$$v_1(y)-v_5(y)=v_1(y)-\bar{\bar{v}}'(y)= Ev_1(y)+Fv_2(y)+Gv_3(y)+Hv_4(y) \quad . \tag{2-225}$$

As in (2-188) we can then obtain:

$$H = \frac{1}{P(y)}\begin{vmatrix} v_1(y) & v_2(y) & v_3(y) & v_1(y)-\bar{\bar{v}}'(y) \\ v_1(y+1) & v_2(y+1) & v_3(y+1) & v_1(y+1)-\bar{\bar{v}}'(y+1) \\ v_1(y+2) & v_2(y+2) & v_3(y+2) & v_1(y+2)-\bar{\bar{v}}'(y+2) \\ v_1(y+3) & v_2(y+3) & v_3(y+3) & v_1(y+3)-\bar{\bar{v}}'(y+3) \end{vmatrix} \quad . \tag{2-226}$$

The procedure that led from (2-188) to (2-194) shows that H can be expressed as follows:

$$\begin{aligned} H = {} & [v_1(y+2)-\bar{\bar{v}}'(y+2)]p_4(y-2)N_4(y-2)+\Big\{[v_1(y+3)-\bar{\bar{v}}'(y+3)]p_4(y-1)+ \\ & +[v_1(y+2)-\bar{\bar{v}}'(y+2)]p_3(y-1)\Big\}N_4(y-1)-\Big\{v_1(y)-\bar{\bar{v}}'(y)+2a[v_1(y+1)- \\ & -\bar{\bar{v}}'(y+1)]\Big\}N_4(y)-[v_1(y+1)-\bar{\bar{v}}'(y+1)]N_4(y+1) \quad . \end{aligned} \tag{2-227}$$

Differentiate (2-194) with respect to y:

$$\begin{aligned} 0 = {} & \bar{\bar{v}}'(y+2)p_4(y-2)N_4(y-2)+\bar{\bar{v}}'(y+3)p_4(y-1)N_4(y-1)+\bar{\bar{v}}'(y+2)p_3(y-1)\cdot \\ & \cdot N_4(y-1)-[\bar{\bar{v}}'(y)+2a\bar{\bar{v}}'(y+1)]N_4(y)-\bar{\bar{v}}'(y+1)N_4(y+1)+\bar{\bar{v}}(y+2)\frac{d}{dy}[p_4(y-2)\cdot \\ & \cdot N_4(y-2)]+\bar{\bar{v}}(y+3)\frac{d}{dy}[p_4(y-1)N_4(y-1)]+\bar{\bar{v}}(y+2)\frac{d}{dy}[p_3(y-1)N_4(y-1)]- \\ & -[\bar{\bar{v}}(y)+2a\bar{\bar{v}}(y+1)][dN_4(y)/dy]-\bar{\bar{v}}(y+1)[dN_4(y+1)/dy] \quad . \end{aligned} \tag{2-228}$$

Add (2-227) and (2-228):

$$H = v_1(y+2)p_4(y-2)N_4(y-2)+[v_1(y+3)p_4(y-1)+v_1(y+2)p_3(y-1)]N_4(y-1)-$$
$$-[v_1(y)+2av_1(y+1)]N_4(y)-v_1(y+1)N_4(y+1)+\bar{\bar{v}}(y+2)\frac{d}{dy}[p_4(y-2)\cdot$$
$$\cdot N_4(y-2)]+\bar{\bar{v}}(y+3)\frac{d}{dy}[p_4(y-1)N_4(y-1)]+\bar{\bar{v}}(y+2)\frac{d}{dy}[p_3(y-1)N_4(y-1)]-$$
$$[\bar{\bar{v}}(y)+2a\bar{\bar{v}}(y+1)][dN_4(y)/dy]-\bar{\bar{v}}(y+1)[dN_4(y+1)/dy] \quad . \qquad (2\text{-}229)$$

Putting y = n+v+1, where n any integer, and observing that:

$$v_1(n+v+1) = B_n = b_{n+2v+1} \quad \text{and} \quad \bar{\bar{v}}(n+v+1) = a_n \ ,$$

we obtain:

$$H = B_{n+2}p_4(n+v-1)N_4(n+v-1)+[B_{n+3}p_4(n+v)+B_{n+2}p_3(n+v)]N_4(n+v)-[B_n+$$
$$+2aB_{n+1}]N_4(n+v+1)-B_{n+1}N_4(n+v+2)+a_{n+2}[p_4'(n+v-1)N_4(n+v-1)+$$
$$+p_4(n+v-1)N_4'(n+v-1)]+a_{n+3}[p_4'(n+v)N_4(n+v)+p_4(n+v)N_4'(n+v)]+$$
$$+a_{n+2}[p_3'(n+v)N_4(n+v)+p_3(n+v)N_4'(n+v)]-[a_n+2a\ a_{n+1}]N_4'(n+v+1)-$$
$$-a_{n+1}N_4'(n+v+2) \quad ,$$

or:

$$H = [B_{n+2}p_4(n+v-1)+a_{n+2}p_4'(n+v-1)]N_4(n+v-1)+[B_{n+3}p_4(n+v)+B_{n+2}\cdot$$
$$\cdot p_3(n+v)+a_{n+3}p_4'(n+v)+a_{n+2}p_3'(n+v)]N_4(n+v)-[B_n+2aB_{n+1}]N_4(n+v+1)-$$
$$-B_{n+1}N_4(n+v+2)+a_{n+2}p_4(n+v-1)N_4'(n+v-1)+[a_{n+3}p_4(n+v)+a_{n+2}p_3(n+v)]\cdot$$
$$\cdot N_4'(n+v)-[a_n+2a\ a_{n+1}]N_4'(n+v+1)-a_{n+1}N_4'(n+v+2) \quad . \qquad (2\text{-}230)$$

Exactly the same expression gives G, if $N_3(y)$ is substituted in place of $N_4(y)$.

An expression for $N'(y)$, at least asymptotic for large $|y|$ in the right half y-plane, can be obtained by direct differentia-

tion of (2-168)-(2-169) or (2-178)-(2-179) (17 pp. 434-461 and pp. 457-459). We obtain, correspondingly:

$$N'_{\substack{4\\3}}(y) = [\pm i\pi/2+\psi(y+v+4)+\psi(y)]N_{\substack{4\\3}}(y)+\frac{(\pm i)^y}{2}\,\frac{\Gamma(y+v+4)\Gamma(y)}{\Gamma(y+v+3\mp ic/2)}\cdot$$

$$\cdot[\psi(y+v+3\mp ic/2)+\frac{d_1\psi(y+v+4\mp ic/2)}{y+v+3\mp ic/2}+$$

$$+\frac{d_2\psi(y+v+5\mp ic/2)}{(y+v+3\mp ic/2)(y+v+4\mp ic/2)}+\ldots\ldots] \qquad (2\text{-}231)$$

$$N'_{\substack{4\\3}}(y) = [\pm i\pi/2+\psi(y+v+4)+\psi(y+3-v)]N_{\substack{4\\3}}(y)+\frac{(\pm i)^y}{2}\,\frac{\Gamma(y+v+4)\Gamma(y+3-v)}{\Gamma(y+6\mp ic/2)}\cdot$$

$$\cdot[\psi(y+6\mp ic/2)+\frac{d_1\psi(y+7\mp ic/2)}{y+6\mp ic/2}+$$

$$+\frac{d_2\psi(y+8\mp ic/2)}{(y+6\mp ic/2)(y+7\mp ic/2)}+\ldots\ldots] \quad , \qquad (2\text{-}232)$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ (17 pp. 241-267). $\psi(z)$ satisfies the relations:

$$\psi(z+1) = \psi(z)+\frac{1}{z} \quad , \quad \psi(1) = -\gamma = \text{Euler's constant} . \qquad (2\text{-}233)$$

It can be checked easily that for real a, b: $G = \overline{H}$. We have expressed H (G) in terms of $N_4(y)$, $N'_4(y)$ (N_3, N'_3) only, just as D (C) was expressed in terms of $N_4(y)$ (N_3).

Concerning the computation of C, D, G, H through equations (2-195) and (2-230) we observe that n can be given values large enough so that the factorial series for N(y) and N'(y) are easily and quickly computed. For such n, $\Gamma(x)$ and $\psi(x)$ can be evaluated with the use of their well-known asymptotic expansions:

$$\Gamma(x) \sim e^{-x}\, x^{x-1/2}\, (2\pi)^{1/2}\Big[1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + \frac{163879}{209018880x^5} + \frac{5246819}{75246796800x^6} - \ldots\ldots\Big] \qquad (2\text{-}234)$$

$$\psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} - \frac{1}{132x^{10}} + \frac{691}{32760x^{12}} - \frac{1}{12x^{14}} + \ldots\ldots\ldots \qquad (2\text{-}235)$$

Even for $x=2$ the second series yields $\psi(2)$ with an accuracy of 6 decimals, while the same is true for $\Gamma(x)$ and $x=3$.

Notice, however, that n can not be given very large values in (2-195), (2-230). Since a_n's and B_n's (the coefficients of the power series in x of $R_1(x)$ and $R_2(x)$) are involved in these relations and the accuracy of their evaluation diminishes as n increases, there is a limitation to the values of n that can be used. It was also observed that for large n, the summation of the terms in the right half sides of (2-195) and (2-230) destroyed the accuracy rapidly by eliminating the first significant decimals of the individual terms. In each particular case there is an optimum range of values of n for which (2-195) and (2-230), with a given accuracy of computation, yield the most accurate results. To make sure that the values of C, D, G, H are the correct ones, one should use (2-195) and (2-230) for about 10 values of n and compare how well the 10 values of these coefficients agree.

As an indication, we give a few results obtained, with 8-decimal accuracy machine computations, in Case I, Chapter 3, PART I. In this case $a=12$, $b=10$, $C=\overline{D}$, $G=\overline{H}$, $R_3(x)=\overline{R}_4(x)$. Besides D and H , the values of $R_2(x)$ for $x=12$ and $x=14$ are given for the first five functions: $v=1,3,5,7,9$. $R_1(x)$ is not given, since it is included in (1-80), defining $R_2(x)$. $x=12$ and $x=14$ fall in the overlapping region between the convergent series (1-80) for $R_2(x)$

and the asymptotic series (2-223): $R_2(x)=e^{-\pi c/4}[GR_3(x)+HR_4(x)]$ yielding values of $R_2(x)$ through G, H and the asymptotic series (1-94) for $R_3(x)$ and $R_4(x)$. Both values of $R_2(x)$ are given for comparison.

v	D	H
1	4.14617-i4.06427	-252.508-i265.563
3	-133.193+i117.365	898.741+i1306.918
5	$(1.24526-i0.965800)\cdot10^4$	$(-0.875955-i4.30989)\cdot10^4$
7	$(-2.32660+i1.5730)\cdot10^6$	$(-2.05026+i3.83348)\cdot10^6$
9	$(7.2577-i4.2497)\cdot10^8$	$(10.0441-i7.42212)\cdot10^8$

v	x	$R_2(x)$ from (1-80)	$R_2(x)$ from (2-2223)
1	12	38.0483	38.1064
	14	114.081	114.082
3	12	-327.000	-326.902
	14	-401.884	-401.875
5	12	$1.46444\cdot10^4$	$1.46278\cdot10^4$
	14	$7.44758\cdot10^3$	$7.44795\cdot10^3$
7	12	$-1.95279\cdot10^6$	$-1.95571\cdot10^6$
	14	$-2.95473\cdot10^5$	$-2.95481\cdot10^5$
9	12	$4.80712\cdot10^8$	$4.80351\cdot10^8$
	14	$-3.75249\cdot10^8$	$-3.75230\cdot10^8$

For x=14, falling roughly in the middle of the overlapping region in this case, the agreement is good up to five significant decimals.

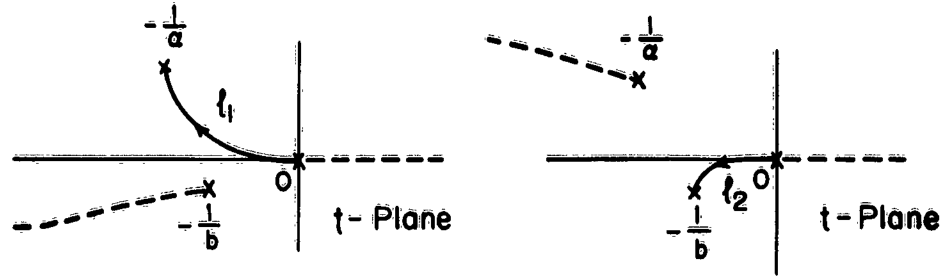


Fig. 2-1 Integration paths ℓ_1, ℓ_2 for $V_{1(y)}, V_{2(y)}$

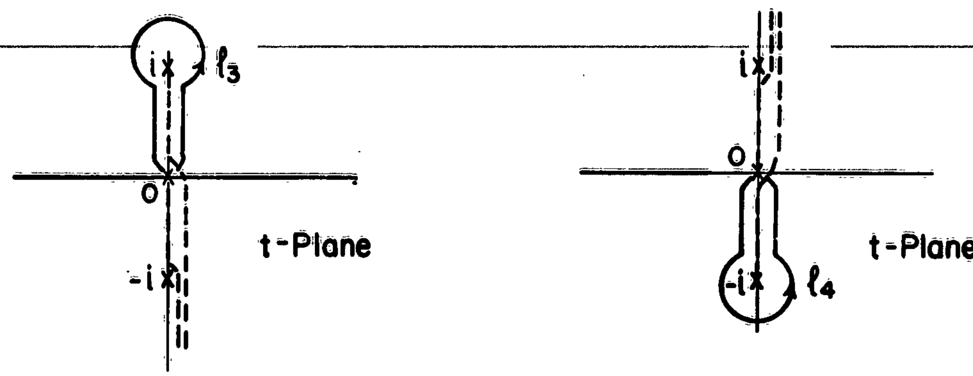


Fig 2-2 Integration paths ℓ_3, ℓ_4 for $V_{3(y)}, V_{4(y)}$.

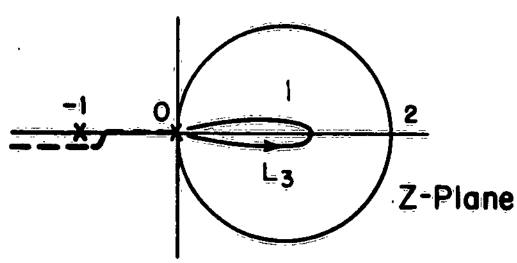


Fig. 2-3 Integration path L_3

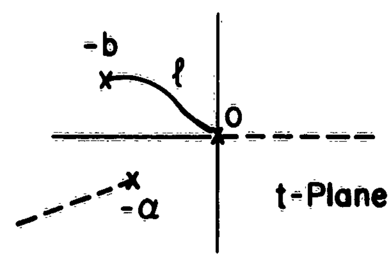


Fig. 2-4 Integration path for $N_{2(y)}$

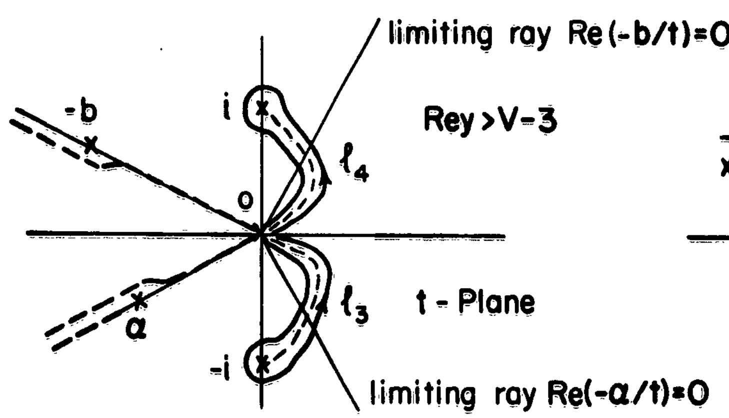


Fig. 2-5 Integration paths ℓ_3, ℓ_4 for $N_{3(y)}, N_{4(y)}$.

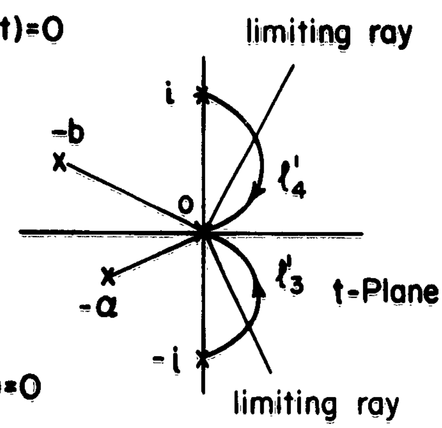


Fig. 2-6 Integration paths ℓ_3', ℓ_4' for $N_{3(y)}, N_{4(y)}$.

CHAPTER 3

REMARKS AND GENERALIZATIONS

The analysis given in the previous chapters led to the complete solution of the problem for the special form: $\varphi(x)=\frac{x+a}{x+b}$ of the stratification function. The complexity of the problem depends exclusively on the form of $\varphi(x)$, since it is this form which determines the number and nature of the singularities of the radial equation (I 1-42). There are three other forms of $\varphi(x)$ for which the method used for $\varphi(x)=\frac{x+a}{x+b}$ can be readily applied. They are given below, together with the differential equation into which (I 1-42) reduces, respectively:

$\varphi(x)=[\frac{x+a}{x+b}]^2$

$$R''(x)+\frac{2(a-b)}{(x+a)(x+b)}R'(x)+[(\frac{x+a}{x+b})^2-\frac{v(v+1)}{x^2}]R(x)=0 \qquad (3\text{-}1)$$

$\varphi(x)=\frac{x^2+a}{x^2+b}$; put: $x^2=z$, $\varphi(z)=\frac{z+a}{z+b}$

$$R''(z)+[\frac{1}{2z}+\frac{a-b}{(z+a)(z+b)}]R'(z)+[\frac{z+a}{4z(z+b)}-\frac{v(v+1)}{4z^2}]R(z)=0 \qquad (3\text{-}2)$$

$\varphi(x)=[\frac{x^2+a}{x^2+b}]^2$; put: $x^2=z$, $\varphi(z)=[\frac{z+a}{z+b}]^2$

$$R''(z)+[\frac{1}{2z}+\frac{2(a-b)}{(z+a)(z+b)}]R'(z)+[\frac{(z+a)^2}{4z(z+b)^2}-\frac{v(v+1)}{4z^2}]R(z)=0 \qquad (3\text{-}3)$$

All the above forms of $\varphi(x)$ represent stratifications similar to $\varphi(x)=\frac{x+a}{x+b}$, as shown in figure (1-1), PART I. The essential point

is that the radial differential equation for TM waves obtained in all cases, has the same number and nature of singularities as (I 1-50). More specifically, three regular singularities at the finite x (or z) plane (one at x=0, or z=0) and an irregular singularity of the first rank at ∞. In the last two cases the solutions around $z=\infty$ become subnormal in nature (9 pp. 417-428 168-171, 10 pp. 63-64), but the modifications required are elementary. The corresponding difference equation (II 2-5) can again be solved along identical lines. Ford's method can be applied equally well to these cases to provide the precise asymptotic expansions required (7 Chapt. VIII). In addition to Ford's Theorems I and VI, used in the preceding analysis, we may have, in the last two cases, to make use of a number of other similar theorems contained in his book. In the last chapter, reference 7, relative examples are included, showing that the method applies without essential modifications. It must be pointed out that the analysis presented in the previous chapters, has modified and generalized Ford's method in two directions. It has provided general expressions for the coefficients C, D and G, H depending only on one solution of the adjoint difference equation (II 2-137). And, mainly, it has dealt successfully with the case of integral values for the difference of exponents (i.e. when $2\nu+1$ is equal to a positive inteder), almost always present, directly or indirectly, in all physical problems. Both these generalizations are applicable in the last two cases (3-2) and (3-3).

The modification of Ford's method, to which we just referred, consists of introducing, in place of x, the variable $y=x+\sigma$ and treat the difference equations (II 2-5) and (II 2-137) in terms of y. The parameter σ is always present in Ford's work (called h) and makes the extension of the analysis to integral values for $2\nu+1$ almost impossible. By introducing y we eliminate this parameter, without rendering the method inapplicable; at the same time

we are able to extend it easily to the case when 2v+1 takes on integral values.

The question that finally arises is whether the analysis developed in the preceding chapters can be applied successfully to more general types of stratification; for instance, to $\varphi(x)$ varying in a manner similar to figure (3-1).

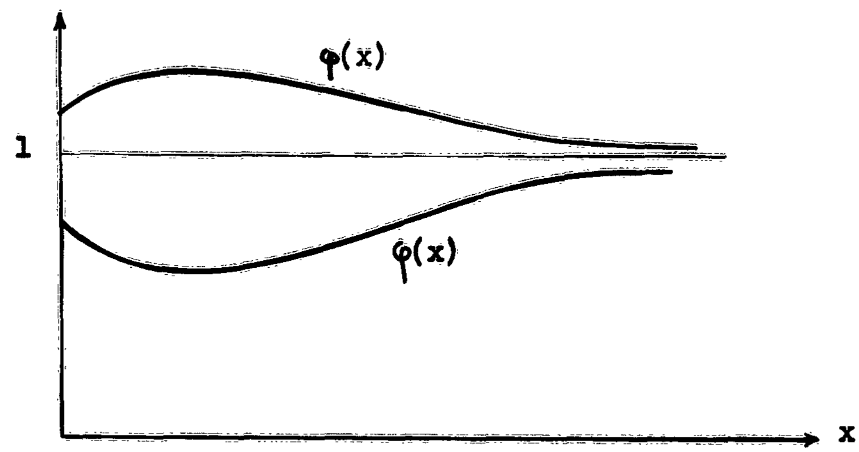


Figure 3-1. More general forms for $\varphi(x)$.

Almost always, (except in trivial cases), the answer depends on the nature and number of singularities that are introduced in equation (I 1-42). The point is illustrated by mentioning certain cases where the method fails. If irregular singularities are introduced in the finite x (or z) plane, then, in spite of the fact that power series solutions can still be found, Ford's theory for obtaining their asymptotic expansions is no longer applicable (7 Chapt. VIII). The difference equation (II 2-5) is no longer normal and its solutions are of such complexity, that no corresponding theorems (like I, VI, or the rest of Ford's theorems) exist, which can be applied to these solutions to yield

the asymptotic expansions of $R_1(x)$, $R_2(x)$. A case like this arises, if $\varphi(x) = [\frac{x+a}{x+b}]^n$, $n \geqslant 3$ and this is the reason why it was not included in the previous list (3-1) to (3-3). It is easily seen that $x=-a$ and $x=-b$ are irregular singular points of (I 1-42).

Another case, where the method fails, arises, if the singularity at $x=\infty$ (or at least $z=\infty$, if we must resort to the change of variable $x=x_{(z)}$), has a rank higher than 1. Normal asymptotic series for $R_3(x)$, $R_4(x)$ can still be obtained, in general, but Ford's theory again fails (7 pp. 339-341), owing to the complexity of the solutions of the difference equation or to the impossibility of even solving it.

Another case arises, if the number of finite regular singularities is large or infinite. For example, $\varphi(x) = 1+ae^{-bx}$, $(b>0)$; the coefficient $\varphi'(x)/\varphi(x)$ of $R'(x)$ in (I 1-42) becomes:

$$\frac{-abe^{-bx}}{1+ae^{-bx}} = -\frac{ab}{a+e^{bx}}$$ and introduces an infinite number of regular singularities, at the zeros of $e^{bx}+a$, in the complex x-plane. Furthermore, in this case, the irregular point $x=\infty$ can not be assigned a finite rank.

In practice such cases can be treated either numerically or by approximating the function $\varphi(x)$ by more simple functions. The numerical results in Chapter 3, PART I, showed that this approximation is valid and permissible. For sharper variations of a more complicated nature, one may divide the interval $0 \leqq x \leqq \infty$ into a finite number of shorter intervals. The problem then is somewhat similar to stratification by layers, requiring additional matching processes at each spherical boundary separating regions of different functional representation for $\varphi(x)$.

If the stratification terminates at a finite distance x, series solutions may be sufficient, as it has already been pointed out; anyway, the problem can be classified as a special case of

stratification by layers.

It must also be pointed out that there are problems where only TE waves are involved. The radial equation (I 1-43) for these waves is more simple than equation (I 1-42) for TM waves. In certain cases, the former equation can be solved, while the latter can not. An example is mentioned in reference 2, but it refers to a finite interval.

With these remarks in mind we can answer the question raised (for problems requiring solution in the whole interval $0 \leqq x \leqq \infty$), as follows: The preceding analysis can be applied to more general stratification functions $\varphi(x)$ as long as:

1) No finite irregular singularities are introduced in equation (I 1-42).
2) The rank of the irregular singularity at x or $z=\infty$ does not become higher than 1; in general, this requirement is more easily satisfied than the preceding one, since in all physical problems, $\varphi(x)$ must reduce to 1, or a finite constant, at x or $z=\infty$.
3) The number of regular singular points introduced in the finite x (or z) plane is reasonably small and permits the use of a bilinear change of variable x (or z) $= \frac{\alpha t+\beta}{\gamma t+\delta}$, which maps the interval of interest in the x-plane (in real cases the real x-axis) into a finite circle around $t=-\beta/\alpha$ in the t-plane, placing all the other singularities of the equation outside this circle. This last restriction is not as fundamental as the first two, but, from the computational point of view, it marks the difference between the possibility of an analytical or completely numerical solution to the problem.

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